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On Quantization with an Indefinite Metric and the Lee Model.

B. FERRETTI

Istituto di Fisica dell'Università - Bologna
Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

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Summary. — The Lee model is discussed again in the case already considered by PAULI and KÄLLÉN *i.e.* when the characteristic equation of the eigen-values of the energy has only real and all differents roots. It is prooved that introducing a suitable redefinition of the ingoing and outgoing states it is possible to construct an unitary S-matrix having reasonable physical properties. The suggested procedure is in some way similar to a renormalization which eliminates the ghost effects.

1. – Introduction.

PAULI and KÄLLÉN [1], studying the Lee model with a cut-off in the interaction, have pointed out that when the coupling constant is greater than a critical value depending from the cut-off, it is no more possible to use a positive definite metric for the quantization of the model and to renormalize with the conventional prescription.

The renormalization is possible only if one uses an indefinite metric. In this case, however, one obviously runs into difficulties with the physical interpretation.

PAULI and KÄLLÉN have examined the case in which the roots of the characteristic equation which determines the eigenvalues of the energy are all real and in the sub-sector 0N, V, all different.

After that, GLASER and KÄLLÉN [2] have taken into consideration the case

(1) G. KÄLLÈN and W. PAULI: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **30**, no. 7 (1955).
(2) V. GLASER and G. KÄLLÉN: *Nucl. Phys.*, **2**, 706 (1956).

of two complex conjugate roots and more recently HEISENBERG [3] has studied the case of two coincident roots. Lately, PAULI, and ASCOLI and MINARDI [4] have again considered the case of the complex conjugate roots.

However, nobody has treated again the case of real distinct roots, as it is generally considered completely settled by the first work of PAULI and KÄLLÉN.

Of course, the interest is not in the Lee model itself, but in the question whether it is possible still to give a physical interpretation in the case of a system for which it is necessary to use an indefinite metric without too drastic change to the usual rules of quantum mechanics. (This aspect of the problem has been particularly stressed by BOGOLJUBOV [5] and by ASCOLI and MINARDI).

The interest of the Lee model lies in the fact that one may try on it general prescriptions which may be invented in order to overcome the difficulties of interpretation of the indefinite metric. It is the purpose of this work to discuss some possibilities in this sense and to show that they may be even applied to the case « without hope » of the two real distinct roots.

Obviously if one wants to maintain the usual relationship between probabilities and state vectors, one cannot use directly the vectors in the « pseudo Hilbert space » with an indefinite metric. Quite generally one will have to establish an homomorphism between the vectors of the pseudo Hilbert space $\mathcal{H}_1$ (with an indefinite metric) and vectors of a Hilbert space $\mathcal{H}_2$ (with a definite positive metric). For the vectors of this Hilbert space it will be possible to apply the usual rules of interpretation.

All the difficulty is in establishing the appropriate homomorphism. Of course, there may be infinite ways to perform this task. In order to be a bit more definite, we shall suppose in general that the homomorphism in question will be a so-called « operator » homomorphism. That is to say it will conserve linear relationships between operators. But even so we have to restrict the field of possibilities with more hypotheses. We shall suppose that our system has an energy operator and that the eigenstates of the energy operator have either positive or negative norm. We shall therefore discard for the moment the case considered by HEISENBERG, in which there are eigenstates with norm equal to 0.

It is then immediately seen that, even disregarding the difficulties put in evidence by PAULI and KÄLLÉN, we cannot consider the possibility of using the full Hilbert space in the usual manner for physical interpretation if we still want to maintain that the (*total*) energy is an observable. In fact, if we

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[3] W. HEISENBERG: *Nucl. Phys.*, **4**, 532 (1957).

[4] W. PAULI: *Proceedings of 1958 Annual International Conference on High Energy Physics at CERN* (1958), p. 127.

[5] N. N. BOGOLJUBOV: *Proceedings of 1958 Annual International Conference on High Energy Physics at CERN* (1958), p. 129.

consider a vector which has a projection different from 0 on an eigenstate of the energy with a negative norm, and if we suppose that this vector represents a state and is, as usual, normalized to 1, we shall immediately find, applying the usual rules of interpretation, that the probability of finding an eigenvalue of the energy corresponding to any eigenstate with positive norm is greater than one. In other words, a measurement of the energy does present the same difficulties that have been pointed out by PAULI and KÄLLÉN for the « scattering experiments ».

This very obvious remark seems to suggest a very simple prescription: « All state vectors corresponding to a physical situation should have component equal to 0 in respect to all eigenstates of the energy of the interacting system with negative norm ». This prescription is obviously compatible with the equation of motion, at least in the case in which the equations of motion are the usual ones.

However, with this prescription alone, one runs again immediately into difficulties in defining the « ingoing » and « outgoing » states necessary for talking of the « scattering experiments ». In fact, as by hypothesis the eigenstates with positive norm of the energy do not form a complete set in the Hilbert space, they are not sufficient in general to express the « ingoing » and « outgoing » states if these states are defined in the usual way.

In order to define the ingoing and outgoing states, it is necessary to make use of the interpretation of the state function in the configuration space and namely to consider the behaviour of the state function amplitude for very great distances. We shall talk in this respect of « asymptotical localization »

If it is possible to establish a 1-1 (isomorphic) correspondence C between the eigenstate of the energy with positive norm and the vectors of a Hilbert space $\mathcal{H}_2$ (of positive definite metric) in which the behaviour of the free particles is described, it might still be possible to define ingoing and outgoing states for our problem if the isomorphic correspondence C does satisfy the following properties:

1) It is a linear correspondence,

2) It leaves invariant scalar products (*).

3) The Hilbert space $\mathcal{H}_2$ is by definition the closure of the linear manifold formed by the vectors corresponding to the eigenvectors of the energy of positive norm (**). It is required that the Hilbert space $\mathcal{H}_2$ so defined be suitable for describing ingoing and outgoing states.

(*) This condition is not strictly necessary for all vectors. It will be sufficient that a similar condition holds only for vectors representing states in which the particles are « asymptotically localized ».

(**) The space so defined has obviously the property of « completeness ».

These conditions are not yet sufficient to allow the definition of an S-matrix and furthermore of an S-matrix having suitable physical properties. It is the particular aim of this paper to show that it is possible to find such correspondence allowing the definition of a sufficiently satisfactory S-matrix, at least in the case of the Lee model, with real distinct roots of the characteristic equation.

At this point we want to point out that the procedure which we have envisaged is perhaps less arbitrary than it might appear at first sight. In fact, for describing experiments, one has to make use essentially of the configuration space (*).

On the other hand, in a relativistic theory one cannot talk of localization of particles at all, but at most of localization of currents and fields. Even in the conventional theory it is not a trivial matter to overcome this difficulty and for this purpose it is necessary to use the concept of the renormalization, etc. But the quantization with the indefinite metric is just interesting because it probably allows to introduce « non-locality » in an essential way. It is therefore conceivable that in a theory with indefinite metrics the concept of « asymptotical localization » has to be reconsidered carefully again and redefined in a suitable manner. That is what is meant by the correspondence C.

For the purpose of this paper, a slightly modified version of the Lee model will be used, which allows to write down explicitly the solution of the eigenstate problem of the energy even in the sector $(\theta, \theta' N, \theta V)$. It will be shown n Appendix II that no essential feature of the Lee model is missed or modified in this way.

The study of this new version will be presented in the second and third paragraph. In the fourth paragraph we shall give an example of the isomorphic correspondence C valid for the modified version of the Lee model, and in the fifth paragraph we shall discuss the possibility of giving a general criterium for constructing such isomorphic correspondence.

2. – «One band» Lee model.

The model which we would like to study in this paragraph will not give rise to the usual difficulty of PAULI and KÄLLÉN for a rather trivial reason. However, it will not be useless to consider this model because it might throw some light on the mechanism of these difficulties and on the manner by which, following the general scheme of the introduction, they might be avoided.

(*) The use of ingoing and outgoing stationary states belonging to the continuum is a schematization which strictly speaking does not correspond by itself to any really possible physical situation, and even mathematically may be objectionable.

As usually, we start with the Hamiltonian of the Lee model in the following form (*)

$$H = H_0 + H_{\text{int}} , \tag{2.1}$$

$$H_0 = m(N^2\psi_V^*\psi_V + \psi_N^*\psi_N) + \sum_k \omega_k a_k^* a_k , \tag{2.1'}$$

$$H_{\text{int}} = -\frac{g}{\sqrt{2\Omega}}\left(\psi_V^*\psi_N \sum \frac{f_i}{\sqrt{\omega_i}} a_{ki} + \sum \frac{f_i}{\sqrt{\omega_i}} a_{ki}^* \psi_N^*\psi_N\right) - \Delta m N^2 \psi_V^*\psi_V . \tag{2.1''}$$

The Hamiltonian is already expressed by means of the renormalized quantities. It is

$$\psi_V = \frac{1}{N}\psi_V \text{ not ren.} , \tag{2.2}$$

$$\{\psi_V\psi_V^*\}_+ = \frac{1}{N^2} , \tag{2.2'}$$

$$(0|\psi_V V)) = 1 , \tag{2.2''}$$

where, with the notation V) we indicate the state in which is present one *physical* V particle, and only one physical V particle. For the other notations, compare with PAULI and KÄLLÉN [1]. f_i indicates a « form factor ». We shall investigate here a model in which f_i will be different from zero only in a narrow neighbourhood of a certain frequency ω. For the purpose of this paragraph we shall assume that f_i is constant and equal to one for $\omega_0 \leqslant \omega_i \leqslant \omega_0 + \Delta\omega$ and that it is zero everywhere else. For other purposes, we shall also investigate the case in which f_i is given by a Gaussian function.

For the reader's convenience, we shall recall here some of the known things about the Lee model and of the results of PAULI and KÄLLÉN which can obviously be applied even to the present model.

We shall use the following notation:

I_V) is a state in which there is a bare V only.

I_N) is a state in which there is a bare N only.

$I_V I_{k_i}$) is a state in which is contained one bare V particle and one θ particle of momentum k_i.

$I_N I_{k_i}$) is a state in which is contained one N particle and one θ particle of momentum k_i only.

etc.

(*) With ψ_V^*, ... we mean the adjoint of ψ_V, ...

Then we have that the state representing a physical V particle is given by

$$V) = C[I_V) + \sum \Phi_i \cdot I_N I_{k_i})] \tag{2.3}$$

(the Φ_i are easily determined but here we are not interested in writing the Φ_i explicitly).

As a consequence of (2.2″) PAULI and KÄLLÉN have proved that $C = N$. One gets also (with the condition that the mass of the physical V is equal to the mass of the N):

$$\Delta m = -\frac{g^2}{2N^2\Omega} \sum \frac{f_i^2}{\omega_i^2}, \tag{2.4}$$

$$N^2 = 1 - \frac{g^2}{2\Omega} \sum \frac{f_i^2}{\omega_i^3}. \tag{2.5}$$

When

$$g^2 > \frac{2\Omega}{\sum (f_i^2/\omega_i^3)},$$

one gets $N^2 < 0$ and then, as PAULI and KÄLLÉN have pointed out, it is necessary to use an indefinite metric. The prescription is to normalize the state to -1 if it contains an odd number of bare V particles.

For the sector (V, θN) one gets the following equation for the eigenvalue of the energy

$$h(z) = z\left(1 - \frac{g^2 z}{2\Omega} \sum \frac{f_i^2}{\omega_i^3 (z - \omega_i)}\right) = 0. \tag{2.6}$$

It is supposed that the system is enclosed in the finite volume Ω; the number of the values of ω_i for which $f_i \neq 0$ is therefore finite. Let it be n. Then $f_i \neq 0$ only for $i = 0, 1, \ldots, n-1$. It is then easy to see that Eq. (2.6) has exactly $n-1$ roots between ω_0 and ω_{n-1}. It has indeed one root (and only one) between ω_{i-1} and ω_i, where i is going from 1 to $n-1$. If g is smaller than g_c there exists one root and only one which is larger than ω_{n-1}, and no root between 0 and ω_0 or smaller than 0.

If g is greater than g_c it exists one root and only one which is smaller than 0, and no root between 0 and ω_0 or greater than ω_{n-1}. In all cases, of course, there exists the root $z = 0$; therefore in all cases Eq. (2.6) has $n+1$ real roots (and only real roots). This result has to be expected for the completeness because in the sector (V, θN) we have just $n+1$ independent states for the not interacting system, namely, one V bare particle state and n(θ, N) free particle states, one for each energy ω_i $(i = 0, \ldots, n-1)$ of the free θ particle.

So in the case of a finite volume in ordinary space and of a finite band in which the θ interacts with the heavy particles, the question of completeness becomes entirely trivial as the Hilbert space which one has to consider has a finite number of dimensions (*). This circumstance makes entirely obvious that already in the section (V, θN) there is some trouble coming from the indefinite metric (if $g > g_c$) if one likes to express, for instance, all the conventionally possible states in which initially a N and a θ particle are present, because it will not be possible to do that, without using also the eigenstate V_-) of the energy with negative eigenvalue.

Now PAULI and KÄLLÉN have shown that in case of $g > g_c$ the eigenstate of the energy corresponding to the negative root of Eq. (2.6) has a negative norm. On the other hand, as already remarked, in the introduction, one cannot allow states with negative norm energy eigenstates components, if the energy is still to be considered observable. It is therefore apparent that already in the section (V, θN) one has to put some restriction on the *physically* possible state of the θ particle if one likes to avoid troubles with the indefinite metric. Under this respect the (V, θN) section is entirely similar to the higher section as for instance the (θV, θθ'N) section.

The essential difference between the (V, θN) section and the higher section is that the restrictions which one has to impose on the possible states do not affect the even conventionally defined scattering states in the (V, θN) sector, but they frequently do affect the conventionally defined scattering states on the higher sectors.

We have now to take into consideration the sector (θV, θθ'N). The eigenstates of the energy for such a sector will be represented by

$$\varphi) = \sum N\Phi_i I_{\nu} I_{k_i}) + \sum \Phi_{sl} I_N I_{ks} I_{kl}) \,. \tag{2.7}$$

This state has to satisfy the equations

$$ZN\Phi_i = (\omega_i - \Delta m)N\Phi_i - \frac{g}{\sqrt{2\Omega}} \cdot \frac{2}{N} \cdot \sum_r \Phi_{ir} \frac{f_r}{\sqrt{\omega_r}} \,, \tag{2.7'}$$

$$(Z - \omega_r - \omega_l)\Phi_{rl} = -\frac{g}{\sqrt{2\Omega}} \cdot \frac{1}{2} \left(\Phi_l \frac{f_r}{\sqrt{\omega_r}} + \Phi_r \frac{f_l}{\sqrt{\omega_l}} \right) . \tag{2.7''}$$

(*) Of course we might also consider the other values of the momentum of the θ-particle for which one has no interaction and which, of course, are infinite in number. However, these states do not enter in any essential way in our consideration (in this case there is a completely trivial 1-1 correspondence between the state of the not-interacting system and the state of the interacting system).

We shall study explicitly the complete system of solutions of Eqs. (2.7′) and (2.7″). This is actually possible with our model if one expands in powers of the parameter $\Delta\omega/\omega_0$. The system of the equations which we have to solve is equivalent to

$$h(z-\omega_i)\Phi_i = \frac{g^2}{2\Omega\sqrt{\omega_i}}\sum_0^{n-1}\frac{\Phi_r}{\sqrt{\omega_r}}\cdot\frac{1}{z-\omega_i-\omega_r}\,. \tag{2.7‴}$$

Let us put

$$\begin{cases} \omega_{l+1} = \omega_0(1+l\varepsilon), \\ z \quad\;\; = \omega_0(1+\xi)\,. \end{cases} \tag{2.8′}$$

and

$$\gamma_l = (1-\xi)\frac{2\Omega\omega_0^2}{g^2}\cdot h(z-\omega_l)\,. \tag{2.8″}$$

We now remark that

$$\frac{g^2}{2\Omega}\cdot\frac{1}{\omega_0^2} \approx \frac{1-N^2}{n} \approx 0\left(\frac{1}{n}\right).$$

We have then to distinguish 3 cases:

$$\text{(A)} \qquad |1-\xi| \gtrsim \frac{1}{n}\,; \qquad z-\omega \neq z_i + 0\,(\Delta\omega)\,,$$

where z_i is any root of Eq. (2.6).

$$\text{(B)} \qquad |1-\xi| \gtrsim \frac{1}{n}\,; \qquad z-\omega_0 = z_i + 0\,(\varepsilon\omega_0)\,,$$

$$\text{(C)} \qquad |1-\xi| \ll \frac{1}{n}\,.$$

Case C) is verified when

$$z-\omega_0 = \omega_0 + 0\,(\varepsilon\omega_0)$$

and it represents the scattering of θ, θ' on a N particle. It never gives rise to any difficulty in our model. Therefore we shall not consider it in detail. We shall have to consider only the cases A) and B). In both cases $|1-\xi| \gtrsim 1/n$ and therefore we might develop in $\Delta\omega/\omega_0$ and, neglecting terms

$0(\Delta\omega^2/\omega_0^2)$ one gets (cf. (2.7‴)):

$$\sum_0^{n-1}{}_l \left\{ 1 - \left(\frac{l}{2} + \frac{1}{1-\xi} \right) (l+s)c + \right.$$
$$\left. + \left[\frac{(s+l)^2}{2(l-\xi)} + \frac{1}{2}(sl^2 + sl^2 + 2sl) + \frac{(1+\xi)^2}{(1-\xi)^2} \right] \varepsilon^2 \right\} \Phi_l = -\gamma_1 \Phi_s .$$

We put now

$$\chi_l = \gamma_l \Phi_l , \qquad c = \frac{1}{2} + \frac{1}{2(1-\xi)} + \frac{1}{(1-\xi)^2} . \tag{2.9}$$

Then, with obvious manipulation, we get easily from the preceding equation:

$$\begin{cases} \chi_s - 2\chi_{s+1} + \chi_{s+2} + 2c\varepsilon^2 \sum \Phi_l = 0 & s = 0, \ldots, n-3, \\ \chi_0 - \chi_1 + \sum_l \left(\frac{1}{2} + \frac{1}{1-\xi} \varepsilon - \left[\frac{2l+1}{2(1-\xi)} + \frac{2l+1}{(1-\xi)^2} + \frac{2l+3}{8} \right] \varepsilon^2 \right) \Phi_l = 0 , \\ \chi_0 + \sum_l \left(1 - \left(\frac{l}{2} + \frac{l}{1-\xi} \right) \varepsilon + \frac{l^2\varepsilon^2}{(1-\xi)^2} + \frac{l^2\varepsilon}{2(1-\xi)} + \frac{3}{8} l^2\varepsilon^2 \right) \Phi_l = 0 . \end{cases} \tag{2.10}$$

From the last equation we get

$$\sum \Phi_l = -\chi_0 + 0\left(\frac{\Delta\omega}{\omega_0} \right) , \tag{2.10'}$$

and therefore, neglecting again terms of the order $0(\Delta\omega^3/\omega_0^3)$ the first $n-2$ equations (2.10) can be written in the following form

$$(\chi_{l+2} - \chi_{l+1}) - (\chi_{l+1} - \chi_l) = 2c\varepsilon^2 \chi_0 .$$

and the system (2.10) is then equivalent to:

$$\begin{cases} \chi_l = \chi_0 (1 + cl(l-1)\varepsilon^2) + l(\chi_1 - \chi_0) , \\ \chi_1 - \chi_0 = \left(\frac{1}{2} + \frac{1}{1-\xi} \right) \varepsilon \sum \Phi_l - \varepsilon^2 \sum \left[\frac{2l+1}{2(1-\xi)} + \ldots \right] \Phi_l . \end{cases} \tag{2.10''}$$

Now, we consider in detail case A). In case A) $z - \omega_l$ differs from the roots of Eq. (2.6) for quantities which are great compared to $\Delta\omega$. Therefore we may develop $1/\gamma_l$ and we may write

$$\frac{1}{\gamma_l} = \frac{1}{\gamma_0} + 0\left(\frac{\Delta\omega}{\omega} \right) . \tag{2.11}$$

Keeping into account Eqs. (2.9) and (2.11) one gets, with the usual approximation from Eq. (2.10′)

$$n/\gamma_0 = -1 .$$

The eigenvalue equation in the case (A) for which Eq. (2.11) is valid, is given (neglecting terms $0(\Delta\omega/\omega)$) by

$$\gamma_0(z-\omega_0) = -\frac{1}{n}. \tag{2.12}$$

It is easy to recognize that Eq. (2.12), together with the condition $(z-\omega_0) \neq z_l + 0(\Delta\omega)$, can be satisfied by two values of $(z-\omega_0)$ only. These values are both negative but greater than z_-. To the smaller of these two values does correspond a negative norm and to the greater one, a positive norm. It remains now to consider simply case (B). In case (B) we have two possibilities:

$$z-\omega_0 = z_- + 0(\Delta\omega) \tag{B_1}$$

(z_- is the energy eigenvalue of the « ghost state » V_-);

$$(z-\omega_0) = 0(\Delta\omega) \tag{B_2}$$

($z-\omega_0 = \omega_0$ is the case (C) .

Let us consider the case (B_1). Let us put

$$\xi = \frac{z_-}{\omega} + \eta\varepsilon = \xi_- + \eta\varepsilon . \tag{2.13}$$

Then:

$$h(z-\omega_l) = h(z_- + \omega_0(\eta - l)\varepsilon) = h'_{z=z_-}\cdot(\eta-l)\omega_0\varepsilon + \tfrac{1}{2}h''_{z=z_-}\cdot(\eta-l)^2\omega_0^2\varepsilon^2 + \ldots$$

and

$$\gamma_l = \overline{\gamma}\varepsilon(\eta-l)(1+\alpha(\eta-l)\varepsilon) + \ldots ,$$

where

$$\overline{\gamma} = (1-\xi_- - \eta\varepsilon)\frac{2\Omega}{g^2}\omega_0^3 h'_{z=z_-} ,$$

$$\alpha = \frac{1}{2}\left(\frac{h''}{h'}\right)_{z=z_-} .$$

Then

$$\frac{1}{\gamma_l} = -\frac{1}{\overline{\gamma}\varepsilon}\cdot\frac{1}{l-\eta} - \frac{\alpha}{\overline{\gamma}} + 0\left(\frac{\Delta\omega}{\omega_0}\right).$$

We may now use this expression, together with Eqs. (2.9) and (2.10″), and we get easily that the equation for the eigenvalues of the energy in case (B_1) might be written in the following form

$$\left(-\frac{1}{\bar{\gamma}}+0\left(\frac{\Delta\omega}{\omega_0}\right)\right)\sum_l \frac{1}{l-\eta}=0\left(\frac{\Delta\omega}{\omega_0}\right). \tag{2.14}$$

Eq. (2.14) in 0-approximation becomes

$$\sum_l \frac{1}{l-\eta}=0\,. \tag{2.14′}$$

Eq. (2.14′) has exactly $n-1$ roots. There are therefore $n-1$ eigenvalues of the energy corresponding to the case (B_1). We shall discuss briefly later the eigenfunction of the energy for case (B_1).

Case (B_2) is completely analogous to case (B_1). Here

$$z-\omega_0=0(\Delta\omega)$$

and using the same kind of development as in case (B_1) we get that there are again $n-1$ eigenvalues of the energy which satisfy the requirements of case (B_2).

We may discuss now the eigenfunctions. Let us write down the form of the eigenfunction for case (B_2). If we put

$$\eta_m = m+\eta_m'\,; \qquad v<\eta_m'<1\,; \qquad m=0,\ldots,\ n-2\,.$$

where η_m is one of the roots of Eq. (2.14′), and if we remember Eqs. (2.7), (2.7′) and (2.7″), we find out easily that these $n-1$ eigenfunctions have the form:

$$\varphi^{(m)}=c\sum_0^{n-1}{}_l \frac{1}{m-l+\eta_m'}\left[I_V I_{k_l})+\frac{g}{\sqrt{2\Omega}}\cdot\frac{1}{\omega_0}\cdot\sum_0^{n-1}{}_r I_N I_{k_l} I_{k_r})\right]+0\left(\frac{\Delta\omega}{\omega_0}\right). \tag{2.15}$$

It is not difficult to show (determining explicitly the state V) in our case) that Eq. (21.5) may even be written in the following form

$$\varphi^{(m)}=\sum_l \frac{c}{m-l+\eta_m'}\,a_{kl}^{*}V)+0\left(\frac{\Delta\omega}{\omega_0}\right). \tag{2.15′}$$

Neglecting the small term $0(\Delta\omega/\omega_0)$ the interpretation of Eq. (2.15′) is particularly simple and clear. Obviously the eigenfunctions of the energy represented by (2.15′) are standing θ particle waves in presence of the physical

V) particle. It is pretty obvious from (2.15′) that the norm of these eigenstates is positive.

Now, in case (B_1), we have instead standing θ particle waves in presence of a « ghost state » V_-) and therefore the eigenstates of the energy corresponding to case (B_1) have a negative norm.

So, summarizing the results obtained until now, we might say that in the cases (A) and (B) together, we have $2n$ eigenstates (n with negative norm and n with positive norm). It is very easy to recognize that in case (C) we have

$$(n-1)(n-2)/2 + (n-1) = n(n-1)/2$$

eigenstates of the energy all with positive norm.

Therefore the total number of eigenstates which we have found is equal to $2n+n(n-1)/2$ which is exactly the number of the corresponding states in the interaction free-case. We have so a check that we have not forgotten any state in our discussion. On the other hand, it is clear that for the completeness we should have to make use of the n states with negative norm. However, as already said, this circumstance does not produce any trouble in the scattering. The reason is due to the fact that in our model there is a gap between the energy of the states with negative norm and the energy of the states with positive norm. This circumstances, merely for the conservation of the energy, will prevent to get any negative norm state in the outgoing wave, if there is not such a state in the ingoing wave.

We might express the same thing even in a different form. Although the eigenstates of the energy of positive norm do not form a complete set, we are able to represent the ingoing and outgoing states by means of the eigenstates of the energy of positive norm only.

Now this is completely trivial in the example which we have discussed because it is automatically made possible by the conservation of the energy. However, it is relevant for the purpose of this paper to consider how actually this thing happens to be verified in the present case.

The state of the system will be in general represented by the superposition:

$$\sum_m \exp[-iE_m t] c_m \varphi^{(m)}) = \sum_{ml} \frac{c_m \exp[-iE_m t]}{m - l + \eta'_m} a^*_{k_l} V) \,. \tag{2.16}$$

We shall have then to consider (cf. also Appendix I) the expression

$$\sum_m \sum_l \frac{c_m}{m - l + \eta'_m} \exp[-iE_m t] \exp[i\boldsymbol{k}_l \boldsymbol{x}] \,,$$

for very great value of $r = |\boldsymbol{x}|$. But remembering Eq. (2.14′) and supposing

that m is not in the neighbourhood either of 0 or of n one gets very easily that

$$(2.16') \qquad \sum_l \frac{\exp[ik_l r]}{m - l + \eta_m^r} \sim -\exp[ik_m r]\left(\pi i + \log\frac{\omega_0 + \Delta\omega - E_m}{\omega_0 - E_m}\right) + 0\left(\frac{1}{r}\right),$$

which is not different from the conventional expression. There is a difference only if m is in the neighbourhood of 0 or n. In these cases one gets also a term which is proportional to $\log r \exp[ik_m r]$. Even this state does not give rise to any inconvenience, provided one takes in the definition of the wave packet (2.16) an interval ΔE_n which is much greater than $\varepsilon\omega_0$. But such condition has to be satisfied anyway if one wants to describe ingoing and outgoing states in a finite volume and, on the other hand, does not put any limit to the precision by which the momentum of the ingoing and outgoing states may be defined, provided, of course, that one takes a sufficiently large volume Ω (and provided it is not asked that the precision is actually infinite). So although the eigenstates of energy of positive norm do not form a complete set and in spite of the fact that we are considering only a Hilbert space of a finite number of dimensions, the eigenstates of the energy of positive norm are in this case perfectly sufficient to describe in a satisfactory manner ingoing and outgoing waves.

Similar considerations will hold by means of a suitable redefinition, even in the case in which it is not automatically granted by the conservation of the energy. Such a case, which is a simple generalization of the model considered in this paragraph, will be discussed in the next paragraph.

3. – Double band model.

We shall consider again in this paragraph a Lee model described by Eqs. (2.1), (2.1′), (2.1″), (2.2), (2.2′) and (2.2″). This time, however, the form factor will be described by

$$(3.1) \qquad \begin{cases} f_i = \exp\left[-\frac{(\omega_i - \omega_1)^2}{\Delta\omega^2}\right] + \exp\left[-\frac{(\omega_i - \omega_2)^2}{\Delta\omega^2}\right], \\ \omega_1 < \omega_2\,; \qquad \frac{\Delta\omega}{\omega_1} \ll 1\,; \qquad \frac{\Delta\omega}{\omega_2 - \omega_1} \ll 1\,. \end{cases}$$

We shall suppose that the constant g is greater than g_c and therefore we shall have to use an indefinite metric. Now let z_- be the energy of the ghost state in the section (V, θN). We suppose

$$(3.2) \qquad \alpha \ll 1\,, \qquad \text{with} \qquad \alpha = -\frac{z_-}{\omega_1}\,.$$

We now make the hypothesis that the coupling constant g is such that the relationships

$$z_- = \omega_1 - \omega_2 - \beta\varepsilon\omega_1 , \tag{3.3}$$

$$0 < \beta < 1 , \tag{3.3'}$$

are satisfied. This hypothesis is made in order that the conservation of the energy may not grant the automatic possibility of describing scattering states in the section $(\theta V, \theta\theta' N)_i$ by means of eigenstates of the energy of positive norm only.

In fact, it will be energetically possible that a θ particle of energy ω_1 colliding with a V particle, is scattered with an energy ω_2 leaving the heavy particle in the ghost state V_-. (Eq. (3.3') is then imposed in order not to have a too sharp resonance for this process. This, however, is not a restrictive condition at all).

Now let us put

$$\begin{cases} \omega_1 l = \omega_1(1 + l\varepsilon) \\ \omega_2 l = \omega_2 + \omega_1 l\varepsilon = \omega_1(1 + \alpha - \beta\varepsilon + l\varepsilon) \end{cases} \tag{3.4}$$

$$f_{1i} = \exp\left[-\frac{(\omega_{1i} - \omega_1)^2}{\Delta\omega^2}\right]; \qquad f_{2i} = \exp\left[-\frac{(\omega_{2i} - \omega_2)^2}{\Delta\omega^2}\right], \quad f_{1i} = f_{2i} = \bar{f}_i , \tag{3.4'}$$

and

$$z = \omega_1(1 + \xi) . \tag{3.5}$$

Of course Eqs. (2.3), (2.4), (2.5) and (2.6) will formally be satisfied even for the case of this paragraph. We might then perform an analysis completely similar to that of the previous paragraph also for the present model. It is then possible to recognize that the interesting case in this section $(\theta V, \theta\theta' N)_i$ is verified when

$$\xi = \frac{\Delta\omega}{\omega_1} . \tag{3.5'}$$

In this case we can write the following equations:

$$h(z - \omega_{1l}) = \omega_1(\xi - l\varepsilon)\left(1 + \frac{g^2\omega_1}{2\Omega}(\xi - l\varepsilon)\sum\frac{f_i^2}{\omega_i^2} + \ldots\right), \tag{3.6}$$

$$h(z - \omega_{1l}) = \omega_1(\xi - l\varepsilon + \beta\varepsilon)h'_{z-z_-} + \frac{\omega_1^2}{2}(\xi - l\varepsilon + \beta\varepsilon)^2 h''_{z-z_-} . \tag{3.6'}$$

We shall now put

$$(3.7)\qquad \begin{cases} \gamma_{1l} = \dfrac{2\Omega\omega_1^3}{g^2\bar{f}_l^2}\, h(z-\omega_{1l}) = \dfrac{2\Omega\omega_1^3}{g^2\bar{f}_l}\,(\xi - l\varepsilon + \ldots)\,, \\ \gamma_{2l} = \dfrac{2\Omega\omega_1^3}{g^2\bar{f}_l^2}\, h'_{z=z_-}(\xi + \beta\varepsilon - l\varepsilon + \ldots)\,, \end{cases}$$

$$\psi_{1l} = \bar{f}_l \Phi_{1l}; \qquad \psi_{2l} = \bar{f}_l \Phi_{2l}\,.$$

Then we get

$$(3.7'')\qquad \begin{cases} \gamma_{1l}\psi_{2l} + \sum\limits_r \psi_{1r}\left(\dfrac{1}{1-\xi} - \dfrac{(l+r)\varepsilon}{(1-\xi)^2} + \ldots\right) + \sum\limits_s \psi_{2s}\left(\dfrac{1}{1+\alpha-\xi} + \ldots\right) = 0\,, \\ \gamma_{2l}\psi_{2l} + \sum\limits_r \psi_{1r}\left(\dfrac{1}{1+\alpha-\xi} + \ldots\right) + \sum\limits_s \psi_{2s}\left(\dfrac{1}{1+2\alpha-\xi} + \ldots\right) = 0\,. \end{cases}$$

Using the same procedure as in the last paragraph, we get:

$$(3.8)\qquad \psi_{1l} = \frac{c_1\bar{f}_l^2}{\xi - l\varepsilon} + \ldots\,; \qquad \psi_{2l} = \frac{c_2\bar{f}_l^2}{\xi - l\varepsilon + \beta\varepsilon}\,.$$

It is then easy to see that, neglecting terms of second order in $\Delta\omega/\omega_1$, the equations which can be used for determining the eigenvalues of the energy are:

$$(3.7''')\qquad \begin{cases} \sum \psi_{1l} + \dfrac{1}{1+\alpha}\sum \psi_{2l} = \sum r\varepsilon\psi_{1r} + \sum \dfrac{r\varepsilon\psi_{2r}}{(1+\alpha)^2} - \gamma_{10}\psi_{10}\,, \\ \sum \psi_{1l} + \dfrac{1+\alpha}{1+2\alpha}\sum \psi_{2l} = \dfrac{1}{1+\alpha}\sum r\varepsilon\psi_{1r} + \dfrac{1+\alpha}{(1+r\alpha)^2}\sum s\varepsilon\psi_{2s} - \gamma_{20}\psi_{20}\,. \end{cases}$$

Now, let us put:

$$(3.9)\qquad \sum \frac{\bar{f}_l^2}{\xi - l\varepsilon} = \sigma_1(\xi)\,,$$

$$(3.9')\qquad \sum \frac{\bar{f}_l^2}{\xi - l\varepsilon + \beta\varepsilon} = \sigma_2(\xi)\,,$$

$$(3.9'')\qquad \sum \bar{f}_l^2 = \varrho\,.$$

Comparing with (3.1), (3.2), and (3.3) and with Eq. (2.6), we get easily:

$$(3.9''')\qquad \varrho = \frac{1}{\varepsilon}\frac{\Delta\omega}{\omega_1} \approx \frac{1}{\alpha}\frac{\omega_1^3}{g^2}\,\Omega\,.$$

Putting:

$$\sum \varphi_{1l}(\xi) = c_1\sigma_1(\xi),$$

$$\sum \varphi_{2l}(\xi) = c_2\sigma_2(\xi).$$

Eq. (3.8) gives (neglecting $\beta\varepsilon$ with respect to ξ):

$$\sum \varepsilon l\psi_{1l} = c_1[\xi\sigma_1(\xi) - \varrho],$$

$$\sum \varepsilon l\psi_{2l} = c_2[\xi\sigma_2(\xi) - \varrho],$$

and keeping into account (3.9‴) and that

$$\gamma_{10} = \frac{2\omega_1^3}{g^2}\Omega\xi, \qquad \gamma_{20} = \frac{2\omega_1^3}{g^2}\Omega h'_{z=z_0}\xi,$$

we get

$$\left.\begin{matrix}\gamma_{10}\\ \gamma_{20}\end{matrix}\right\} \sim \alpha\varrho\cdot 0\left(\frac{\Delta\omega}{\omega_1}\right). \tag{3.10}$$

Therefore the system of equations of the order 0 (neglecting terms $0(\Delta\omega/\omega_1)$) which determine the eigenvalue of the energy is then

$$\begin{cases} C_1\sigma_1 + \dfrac{1}{1+\alpha}\, C_2\sigma_2 = -\varrho\left[C_1 + \dfrac{1}{(1+\alpha)^2}\, C_2\right], \\ C_1\sigma_1 + \dfrac{1+\alpha}{1+2\alpha}\, C_2\sigma_2 = -\varrho\left[\dfrac{C_1}{1+\alpha} + \dfrac{1+\alpha}{(1+2\alpha)^2}\, C_2\right]. \end{cases} \tag{3.11}$$

We shall see later that when

$$\sigma_1 \approx \varrho,$$

then

$$\frac{\sigma_1}{\sigma_2} \approx \frac{\Delta\omega}{\omega_1}. \tag{3.11′}$$

Eqs. (3.11) keeping into account (3.11′) are then easily transformed into

$$\left(\frac{(1+\alpha)^2}{1+2\alpha} - 1\right)\sigma_1 = \varrho\left(\frac{1}{1+\alpha} - \frac{(1+\alpha)^2}{1+2\alpha}\right) \approx \frac{1}{2}\left(\frac{1}{1+\alpha} - \frac{(1+\alpha)^2}{1+2\alpha}\right)\frac{\Delta\omega}{\omega_1}, \tag{3.11″}$$

and remembering that

$$\sigma_1 = \sum \frac{\bar{f}_l^2}{\xi - l\varepsilon} = \frac{1}{2}\sum \frac{\bar{f}_l^2}{m - l + \eta},$$

we get the following energy eigenvalue characteristic equation in the approximation zero:

$$\sum \frac{\bar{f}_l^2}{m-l+\eta} = 0\,; \qquad 0<\eta<1\,. \tag{3.11'''}$$

Subtracting then the two equations (11), and neglecting terms of the order $\Delta\omega/\omega_1$, one gets

$$C_2\sigma_2\left(\frac{1}{1+\alpha}-\frac{1+\alpha}{1+2\alpha}\right) = -\varrho C_1\left(1-\frac{1}{1+\alpha}\right), \tag{3.12}$$

and that shows that when (3.11′) is satisfied

$$\frac{C_2}{C_1} \approx 0\left(\frac{\varrho}{\sigma_2}\right) \approx 0\left(\frac{\sigma_1}{\sigma_2}\right) \approx 0\left(\frac{\Delta\omega}{\omega_1}\right), \tag{3.12'}$$

which means that C_2 is negligible with respect to C_1 in the approximation zero.

Now we have really to show that (3.11′) is satisfied. For this purpose we have simply to prove that when the condition of (3.11″) is satisfied, *i.e.* when (3.11‴) is satisfied, then no matter which m is,

$$\sum \frac{\bar{f}_l^2}{m+\eta_m+\beta-l} \neq 0\,. \tag{3.13}$$

This again is equivalent to showing that

$$\sum_l \left(\frac{\bar{f}_{m+l}^2}{l-\eta_m-\beta}-\frac{\bar{f}_{m+l}^2}{l-\eta_m}\right) \neq 0\,.$$

Now we have to consider two cases.

1st case:

$$\eta_m+\beta<1\,,$$

then:

$$\sum_l \left(l\,\frac{\bar{f}_{m+l}^2}{l-\eta_m-\beta}-\frac{\bar{f}_{m+l}^2}{l-\eta_m}\right) = \beta\sum_{-\infty}^{+m}\frac{\bar{f}_{m+l}^2}{(l-\eta_m-\beta)(l-\eta_m)}\,.$$

Now the sum which appears in the second member has all the terms greater than 0 because $l-\eta-\beta$ and $l-\eta$ always have the same sign. We have therefore proved (3.13) in the first case.

2nd case

$$\eta_m+\beta = 1+\beta'; \qquad 0<\beta'<\eta_m\,.$$

In this case:

$$\sum\left(\frac{\bar{f}^2_{m+l}}{l-\eta_m-\beta}-\frac{\bar{f}^2_{m+l}}{l-\eta_m}\right)=-(\eta_m-\beta')\sum\frac{\bar{f}^2_{m+l}}{(l-\beta')(l-\eta)}+\frac{\sum\bar{f}^2_{m+l}-\bar{f}^2_{m+l}}{l-1-\beta'}.$$

But

$$\sum\frac{\bar{f}^2_{m+l}-\bar{f}^2_{m+l-1}}{l-1-\beta'}=0(\varepsilon\sigma_2),\qquad \eta_m-\beta'=1-\beta\neq 0,$$

therefore (3.13) holds. We may remark that if $\eta+\beta\to 1$, $\sum\bar{f}^2/(l-\eta-\beta)\to$ $\to\pm\infty$; in this case (3.13) holds *a fortiori*.

We remark also that $\sum\bar{f}^2_{l+m}/(l-\beta')(l-\eta)$ is always necessarily of the order of unity, even it m is very big and consequently $\bar{f}^2_m$ is very small because of the eigenvalue equation (3.11′). (η_m must be so small that there are terms of 0(1) in the sum).

Therefore (3.11′) is completely proved.

It does exist of course an analogous solution in which, however,

$$\frac{C_1}{C_2}\approx 0\left(\frac{\Delta\omega}{\omega_1}\right), \tag{3.12″}$$

and the characteristic equation of the eigenvalue energy is in this case

$$\sum\frac{\bar{f}^2_l}{m+\eta'_m+\beta-l}=0. \tag{3.11‴}$$

In the first case when (3.12′) and (3.11‴) are valid, it is easy to recognize that the eigenstate of the energy may be written under the form

$$C_1\sum\frac{\bar{f}_l}{m+\eta_m-l}a^*_l V)+C_2\sum\frac{\bar{f}_l}{m+\eta_m+\beta-l}a^*_k V_-), \tag{3.14}$$

plus « other terms ».

The « other terms » which appear in Eq. (3.14) are of the type scattering of « θ′ and θ on N » and therefore they are of positive norm and cannot give rise to any trouble. It is easy to see that the state represented by Eq. (3.14) is of positive norm. In the approximation 0 in which one neglects all terms of the order $\Delta\omega/\omega_1$ it would simply represent a standing θ wave in presence of a particle. Therefore even in the present case in the approximation 0 we are exactly in the same situation in which we were with the model of the previous paragraph, and, neglecting terms of the order of $\Delta\omega/\omega_1$ as will be seen in more detail in the next paragraph, it would be possible to represent

ingoing or outgoing θ particles packets in presence of a V particle, without making use of the eigenstate of the energy of negative norm. Unfortunately this is not any more the situation when one is taking into account the term of the order $\Delta\omega/\omega_1$

$$C_2 \sum \frac{\bar{f}_l}{m + \eta_m + \beta - l} a_l^* V_-) ,$$

which is present in Eq. (3.14). In fact, a superposition of eigenstates of energy of positive norm of the form (3.14) will always contain not only a θ wave packet in presence of V, but also a component in which a ghost V_- appears. In order to eliminate this ghost V_-, for instance in the ingoing wave, to obtain a conventional expression of a θ ingoing in presence of a V particle only, one should make use of eigenstates of the energy of negative norm which are obtained when (3.12″) and (3.11‴) are valid instead of (3.12′) and (3.11‴) and then, of course, the ghost would appear in the outgoing waves.

We shall see in the next paragraph how this difficulty may be overcome, following the general prescription which we have outlined in the introduction.

4. – Construction of an unitary S matrix for the double band model.

The purpose of this paragraph is to show that it is possible to find a correspondence C between the eigenstates of positive norm of the energy of the Lee model considered in the previous paragraph, and the vectors of another Hilbert space suitable for describing incoming and outgoing states, and to show that by means of this correspondence it is possible to define an S-matrix with reasonable physical properties.

For this purpose we remind that there is no difficulty for the eigenstates of the Hamiltonian representing scattering of two θ particles with and N particle, because in our model the energy of those eigenstates does not overlap with the energy of the states of negative norm. As already pointed out, the only states which in our model will give rise to trouble are the eigenstates represented by Eq. (3.14).

In the approximation 0 (with respect to an expansion in $\Delta\omega/\omega_1$) the states (3.14) may be represented by:

(4.1) $$\gamma^{(m)}) = \sum_l \beta_{ml} a_l^* V) ,$$

where

(4.1′) $$\beta_{ml} = \frac{C_m \bar{f}_l}{m + \eta_m - l} .$$

We now remark that the set of states

$$\theta^{(m)} = \sum_l \beta_{ml} a_l^* 0) \tag{4.1''}$$

is exactly an orthogonal set. In fact:

$$\sum_l \beta_{nl}^* \beta_{ml} = C_{1n}^* C_{1m} \sum_l \frac{\bar{f}_l^2}{(n+\eta_n-l)} \frac{1}{(m+\eta_m-l)} =$$

$$= \frac{C_{1n}^* C_{1m}}{m-n+\eta_m-\eta_n} \left(\sum_l \frac{\bar{f}_l^2}{n+\eta_n-l} - \sum_l \frac{\bar{f}_l^2}{m+\eta_m-l} \right),$$

but $\sum \bar{f}_l^2/(n+\eta_n-l)$ and $\sum \bar{f}_l^2/(m+\eta_m-l)$ are both equal to zero because of Eq. (3.11''''), and therefore the scalar product $(\theta^{(n)}|\theta^{(m)})$ vanishes. (The denominator $m-n+\eta_m-\eta_n$, is obviously different from 0 for $m \neq n$ because η_m and η_n are both positive numbers less than 1.)

In order to construct the correspondence C we first put in correspondence the state V) representing a physical V particle with a state V) of $\mathcal{H}_2$ representing a not-interacting V particle. (It should be noted that the state V_0) is *not* the state 1_r) representing a bare V particle. The state V_0) is of course taken with a positive norm.) Then we put in correspondence the states $\gamma^{(m)}$) to the states

$$\beta_m) = \sum \beta_{ml} a_l^* V_0) \tag{4.2}$$

of $\mathcal{H}_2$. It has to be noted that no inconvenience does arise from the substitution $V) \to V_0)$ if we limit ourselves to consider in the second Hilbert space only « asymptotical » θ packets, *i.e.*, θ packets which do not overlap in the ordinary space with the physical V particle (cf. Appendix).

Now, as $V_0)$ is not interacting with θ the set of vectors $\beta_m)$ is orthogonal as a consequence of the fact that the vectors defined by (4.1'') are orthogonal, as already proved. Therefore the correspondence which we are now defining is satisfying the requirement (4.2) of the introduction for the correspondence C.

The extension of the correspondence C to the other eigenstates of positive norm is trivial and not interesting. Let us call K the unitary operator which performs the correspondence C between Hilbert space 1 and Hilbert space 2. We remark that K does not depend on the time. We have now to verify that the operator K defined in this manner allows the construction of a suitable S-matrix. For this purpose let us consider the two operators:

$$U = \exp[-iHt], \tag{4.3}$$

$$U_0 = \exp[-iH_0 t], \tag{4.3'}$$

where H is the total Hamiltonian and H_0 is the non-interacting Hamiltonian. U will operate only on the subspace of the eigenstates of H of positive norm. In this subspace U is unitary (not pseudounitary).

We have now to consider

$$U_1(t', t) = U_0^{-1}(t') K U(t' - t) K^{-1} U_0(t) . \tag{4.4}$$

Then, if the limit

$$S = \lim_{t' \to \infty,\ t \to \infty} U_1(t', t) , \tag{4.5}$$

does exist, this will be by definition the S-matrix.

We have now to prove that the limit (4.5) does really exist and that the S-matrix (4.5) is a physically reasonable S-matrix. For this purpose let us put

$$U^{(0)} = K U K^{-1} . \tag{4.4'}$$

Obviously $U^{(0)}$ is obtained from U replacing in place of the eigenvectors (4.1), the vectors (4.2). Then

$$U_1(t', t) = U_0^{-1}(t')\, U^{(0)}(t' - t)\, U_0(t) . \tag{4.4''}$$

Now keeping into account very well known properties of the Hilbert space (*), the proof of the existence of the limit (4.5) is then equivalent to the proof of the existence of the limit (**):

$$\lim_{t \to \infty,\ t' > t} U_1(t', t) = 1 . \tag{4.5'}$$

We shall prove that a relation of the type (4.5') holds if the operator U_1 is applied to wave packets

$$\psi) = \sum_m c_m \exp\left[- i E_m t\right] \beta_m) , \tag{4.6}$$

where the coefficient c_m in the limit of the very large volume may be prolongated in an analytic function $c(m)$ of the energy which is limited in a neighbourhood as small as we like—but finite—of the real axis (***).

(*) « Completeness » of the Hilbert space $\mathcal{H}_2$ (see the remark p. 395 in the introduction).

(**) Of course the existence of the limit $\lim_{t' \to -\infty,\ t < t'} U_1(t', t) = 1$ is also necessary but follows in the same way as (4.5').

(***) Such a hypothesis is merely made in order to define conveniently the « very future » and « very past » time which appears in the limit (4.5). It will be noted that with the passage of the time the coefficient c_m will go over to $c_m \exp\left[- i E_m t\right]$ and, due to the exponential factor, for t sufficiently great in absolute value, the analytical prolongation of the coefficient will become in modulus bigger than any prefixed number in any finite neighbourhood of the real axis.

Furthermore, we suppose that c_m near 0 may be put equal to mG_m with G_m which may be prolongated in an analytical function limited in a neighbourhood of 0.

Now let us put

$$\alpha_k) = a_k^* V_0) \tag{4.7}$$

then

$$\beta_m) = \sum_k \beta_{mk}\alpha_k) \,. \tag{4.2'}$$

It follows (cf. (4.6))

$$U_1(t', t'')\psi) = \sum_k \sum_m \exp[-iE_m t']c_m\beta_{mk} \exp[i\omega_k(t'-t)]\alpha_k) \,.$$

In order to verify (4.5') we have not to verify that:

$$\lim_{t \to \infty} \sum_k |\sum_m (c_m\beta_{mk} \exp[-iE_m t' + i\omega_k(t'-t)] - c_m\beta_{mk} \exp[-iE_m t])|^2 = 0 \,. \tag{4.5''}$$

For this purpose we remark that we may put in the limit $\Omega \to \infty$

$$\sum_m c_m\beta_{mk} \exp[-iE_m t](\exp[-i(t_m - \omega_k)(t'-t)] - 1) =$$
$$= c_k B_k(\exp[-i(E_k - \omega_k)(t'-t)] - 1) +$$
$$+ \int_C c_m\beta_{mk} \exp[-iE_m t](\exp[-i(E_m - \omega_k)t'-t)] - 1) \, \mathrm{d}m \,,$$

where we have performed a transformation to the continuous variable m and the contour C has a indenting under the pole $m = k$ of β_{mk}.

In fact, remembering (4.1') we have to calculate an expression like $\sum g_m/(m + \eta_m - k)$, where g_m is regular in a neighbourhood of k. This expression for $\Omega \to \infty$ may always be written as

$$\int_0^{k-\varepsilon_k'} g_m \frac{\mathrm{d}m}{m-k} + \int_{k+\varepsilon_k''}^{\infty} g_m \frac{\mathrm{d}m}{m-k} + \text{terms } 0\left(\frac{1}{\Omega^{\frac{1}{2}}}\right),$$

where ε_k' and ε_k'' do depend only on η_m and not on g_m. But:

$$\int_0^{k-\varepsilon_k'} \frac{g_m \mathrm{d}m}{m-k} + \int_{k+\varepsilon_k''}^{\infty} \frac{g_m \mathrm{d}m}{m-k} = P\int_0^{\infty} \frac{g_m \mathrm{d}m}{m-k} + g_k \lg \frac{\varepsilon_k''}{\varepsilon_k'} = \int_C \frac{g_m \mathrm{d}m}{m-k} + g_k\left(\lg \frac{\varepsilon_k''}{\varepsilon_k'} - i\pi\right),$$

and therefore our statement is verified.

B_k is a function of k which may even diverge for $k \to \infty$. We however, always assume that $c_k B_k \to 0$ for $k \to 0$ with a sufficient velocity. This is always true, if we consider suitable wave packets for which the spread in momentum is small. On the other hand, if $\Omega = L^3$ for $\Omega \to \infty$ we may always have that

$$\frac{t'-t}{L} \to 0 ;$$

this condition is of course necessary in order to be able to talk of scattering in a box at all, and is not a limitation on t' but a condition on L. Therefore

$$\lim_{\Omega\to\infty} \exp[i(E_k - \omega_k)(t'-t)] - 1 \to 0 .$$

We have then to consider only:

$$\int_c \mathrm{d}m\, c_m \beta_{mk} \exp[-iE_m t](\exp[-i(E_m - \omega_k)(t'-t)] - 1) .$$

Now let us put

$$\zeta_{\overline{m},\tau}) = \sum_{\overline{m}}^{\infty} c_m \exp[-iE_m\tau]\beta_m) . \tag{4.8}$$

We may always choose $\overline{m}$ in such a way that

$$|\zeta_{\overline{m},\tau})|^2 < \varepsilon^2 , \tag{4.8'}$$

ε being any positive number. Then

$$\sum_k |\sum_{\overline{m}}^{\infty} c_m \beta_{mk} \exp[-iE_m\tau] \exp[-i\omega_k\tau']|^2 < \varepsilon^2 ,$$

because the previous expression is the modulus square of the vector $\zeta_{\overline{m},\tau})$ (this modulus has been written by means of the components of $\zeta_{\overline{m},\tau})$ with respect to the axis $\exp[i\omega\tau'] \cdot \alpha_k)$). This is true no matter what τ and τ' are because $|\zeta_{\overline{m},\tau}|^2$ is independent of τ and τ'. We put now:

$$\psi_{1\overline{m}}) = \sum_k \sum_0^{\overline{m}} c_m \beta_{mk} \exp[-iE_m t] \exp[-i(E_m - \omega_k)t'] \alpha_k) ,$$

$$\zeta_{1\overline{m}}) = \sum_k \sum_{\overline{m}}^{\infty} c_m \beta_{mk} \exp[-iE_m t] \exp[-i(E_m - \omega_k)\tau'] \alpha_k) ,$$

$$\psi_{2\overline{m}}) = \sum_k \sum_0^{\overline{m}} c_m \beta_{mk} \exp[-iE_m t] \alpha_k) ,$$

$$\zeta_{2\overline{m}}) = \sum_k \sum_{\overline{m}}^{\infty} c_m \beta_{mk} \exp[-iE_m t] \alpha_k) .$$

We have to determine:

$$P^2 = |\psi_{1\overline{m}}) + \zeta_{1\overline{m}}) - \varphi_{2\overline{m}}) - \zeta_{2\overline{m}})|^2$$

and then this quantity satisfies the following inequality

$$(4.8'') \qquad P^2 \leqslant |\psi_{1\overline{m}}) - \psi_{2\overline{m}})|^2 + 2\,|\psi_{1\overline{m}}) - \psi_{2\overline{m}})|\cdot|\zeta_{1\overline{m}}) - \zeta_{2\overline{m}})| + \\ + |\zeta_{1\overline{m}}) - \zeta_{2\overline{m}})|^2 \leqslant (\psi_{1\overline{m}}) - \psi_{2\overline{m}})|^2 + 8\varepsilon + 4\varepsilon^2,$$

(because $|\psi)_{1\overline{m}}|^2 \leqslant 1$ and $|\psi_{2\overline{m}})|^2 \leqslant 1$).

In order to prove that $P^2 \to 0$ when $t \to \infty$ it will be then sufficient to prove that $|\psi_{1\overline{m}}) - \psi_{2\overline{m}})|^2 \to 0$ for *any* finite $\overline{m}$.

Of course (4.8″) holds *a fortiori* even if we do not extend the sum with respect to the index k to all the interval of variability of k and it holds therefore even if we break the whole sum with respect to k into two parts for each part of the sum separately.

Now let us suppose that $\overline{m}$ is such that Eq. (4.8″) holds. We shall break then the sum with respect to the index k into two parts. We shall indeed consider:

$$P_1^2 = \int_0^{2\overline{m}} \mathrm{d}k \left| \int_{C_1} \mathrm{d}m\, c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)(t' - t)] - 1) \right|^2,$$

and

$$P_2^2 = \int_{2\overline{m}}^{\infty} \mathrm{d}k \left| \int_0^{\overline{m}} \mathrm{d}m\, c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)(t' - t)] - 1) \right|^2,$$

where C_1 is going (with the indenting under $m = k$) from 0 to $3\overline{m}$.

For reasons already discussed, it will be enough to show that P_1^2 and P_2^2 are going to 0 when t is going to infinity for any $\overline{m}$. For this purpose, beginning with P_1^2 we change the contour of integration C_1 in such a way that now it will be composed of three rectilinear segments, one going from 0 to $-i\Delta m$, the other going from $-i\Delta m$ to $3\overline{m} - i\Delta m$ and the third one from $3\overline{m} - i\Delta m$ to $3\overline{m}$.

Now in the first interval $0\text{-}i\Delta m$ we have $c_m < mG_0$ (*) and therefore the absolute value of the integral in the first interval is less than $2C_1(\exp[\Delta m^2/\Delta\omega^2])G_0\Delta m$; *i.e.*, of a quantity as small as we like if we take Δm small enough.

(*) Remember the hypothesis about c_m.

In the third interval

$$|\beta_{mk}| \leqslant C_1 \frac{\exp\left[(4\Delta m\overline{m} + \Delta m^2)/\Delta\omega^2\right]}{\overline{m}} = \frac{\overline{C}_1}{\overline{m}}.$$

Therefore the absolute value of the integral in the third interval becomes smaller than $\overline{C}_1 M(\Delta m/\overline{m})$, where M (*) is a fixed number and is greater than the upper limit of the function c_M in the neighbourhood of the real axis between the two parallels to the real axis at a distance $\geqslant$ than Δm from the real axis. Therefore again this value may become as small as we like, provided we take Δm sufficiently small.

Now, on the other hand, the absolute value of the integral on the second interval will be smaller than

$$\overline{c}_1 M \exp\left[-\frac{\Delta m^2}{2\mu} t\right] \cdot \frac{3\overline{m}}{\Delta m}, \quad (*)$$

because

$$|\beta_{mk}| \leqslant \frac{\overline{c}_1}{\Delta m}.$$

It will also become as small as we like if we take t sufficiently great independently of $\overline{m}$ and Δm. So the absolute value of the first integrals with respect to m will become smaller than any prefixed ε_1^2 and P_1^2 will be then smaller than $2\varepsilon_1^2$.

Now we discuss P_2^2. We evaluate the integral with respect to m along the contour $0 \to -i\Delta m$; $-i\Delta m \to \overline{m} - i\Delta m$; $\overline{m} - i\Delta m \to \overline{m}$. It is then easy to see that in the first and third intervals the integral will be in absolute value less than $\overline{C}_1 M(\Delta m/(k - \overline{m}))$ and in the second interval smaller than

$$\overline{C}_1 M \frac{\exp\left[-(\Delta m^2/2\mu)t\right]}{k - \overline{m}}.$$

Therefore again we may have

$$P_2^2 < \int_{2\overline{m}}^{\infty} \mathrm{d}k \frac{\varepsilon_2^2}{|k - m|^2} = \frac{\varepsilon_2^2}{\overline{m}},$$

with ε_2 being any positive number.

We have proved the relation (4.5″) and therefore that the limit (4.5) does exist. We may now add some remarks. The proof of the existence of the

(*) For $\Delta m \leqslant 2\mu$ where μ is the mass of the θ-particle.

limit (4.5) is in itself a proof that the θV wave packet will behave in the limit of an extremely great box, and for sufficiently far past time or far future time in a physically reasonable manner. In fact, we have proved that for such times this wave packet behaves as nearly as a free wave packet as we like. In order to complete the proof that the formalism which we are suggesting here gives, at least for the case of the model which we are discussing, physical results which are as reasonable as possible, we have simply to show that no unreasonable things happen for times which are not in the very far past or in the very far future.

For this purpose we remark that the representation in the configuration space of the vectors β_m) has a correct asymptotic behaviour (for $r \to \infty$). That means that it behaves like

$$\beta' \frac{\exp[-imr]}{r} + \beta'' \frac{\exp[imr]}{r}.$$

The proof that for not very far time, the representation of ψ) in configuration space will vanish for very high value of r (which is all we need to know) is then entirely analogous and even simpler than the proof of the limit (4.5). The only difference is that this time one has to play with the variable r instead of playing with the variable t. This remark achieves the proof that in our case a reasonable correspondence C between the $\mathcal{H}_1$ and $\mathcal{H}_2$ does exist.

Of course, it is pretty obvious that from the existence of one correspondence C it follows that an infinite number of such correspondences exist. Therefore the method is not unambiguous. We shall discuss in the conclusion the probable meaning of this ambiguity.

5. – Extension of the method to a more general model.

We have now to see further that the possibility of constructing a suitable correspondence C is not an accidental case for the particular model which we have considered. In fact, we shall now briefly outline a procedure which may be applied quite in general to a Lee model without any particular restriction on the function f_i. For this purpose let us consider a Lee model with a quite general form factor f_i. We shall suppose for the moment only that the form factor f_i will vanish for arguments greater than a certain argument, which may be as big as we like.

Furthermore of course we shall suppose that f_i is limited every where. Then the critical value g_c of the coupling constant will be finite. We remark now that there is a second value g_c' greater than the g_c such that, if the coupling

constant satisfies the inequalities

$$g_c < g < g'_c \,, \tag{5.1}$$

there is a gap between the spectrum of the energy of the eigenstates of the total Hamiltonian with positive norm and the spectrum of the energy of the eigenstate of the Hamiltonian with negative norm (we consider only the states belonging to the sectors V, Nθ and the sector θV Nθ θ'). In fact, the energy of the ghost states V_- approaches $-\infty$ when g approaches g_c from the above. Remembering that by hypothesis f_i vanishes for values of the argument greater than a certain value, it is then easy to verify the previous statement.

Then, in this case, for values of the coupling constant satisfying the inequalities (5.1), there is no difficulty whatsoever; the correspondence C may just be the identity because we are essentially in the same case considered in Section 2 and conservation of the energy alone will grant the unitarity of the S-matrix. But, we want of course to consider the case in which g is not only greater than g_c but also of g'_c. In this case C cannot be any more the identity and we have to find out a procedure for constructing C in a suitable manner. For this purpose we shall start from a value g of the coupling constant satisfying the inequality (5.1) and we shall write the total Hamiltonian for the value $g' > g'_c$ in the following way:

$$H = H_0 + H_1 + H_2 \,, \tag{5.2}$$

where

$$H_1 = -\frac{g}{\sqrt{2\Omega}} \frac{N_{g'}}{N_g} \left(\psi^*_{\mathrm{V}} \psi_{\mathrm{N}} \sum \frac{f_i}{\sqrt{\omega_i}} a_{k_i} + \sum \frac{f_i}{\sqrt{\omega_i}} a^*_{k_i} \psi^*_{\mathrm{N}} \psi_{\mathrm{V}} \right) - \Delta m_g N^2_{g'} \psi^*_{\mathrm{V}} \psi^*_{\mathrm{V}} \,, \tag{5.2'}$$

$$H_2 = -\frac{\Delta g}{\sqrt{2\Omega}} \left(\psi^*_{\mathrm{V}} \psi_{\mathrm{N}} \sum \frac{f_i}{\sqrt{\omega_i}} a_{k_i} + \sum \frac{f_i}{\sqrt{\omega_i}} a^*_{k_i} \psi^*_{\mathrm{N}} \psi_{\mathrm{V}} \right) - (\Delta m_{g'} - \Delta m_g) N^2_{g'} \psi^*_{\mathrm{V}} \psi_{\mathrm{V}} \,, \tag{5.2''}$$

N'_g and N_g are the renormalization constant for g' and g. The operator ψ_{V} is renormalized relatively to g' and

$$\Delta g = g' - \frac{N_{g'}}{N_g} \cdot g \,.$$

The mass renormalization constants Δmg and $\Delta mg'$ will be fixed in such a way that the mass of the physical V) particle does not change from passing from the value g to the value g'.

Now we remark that in general if the Hamiltonian can be written under

the form (2), if we put

$$U = \exp[-iHt]$$

$$U_1 = \exp[-i(H_0 + H_1)t]$$

$$U_0 = \exp[-iH_0 t]$$

and if

$$S_0 = \lim_{t' \to \infty,\, t \to -\infty} U_0^{-1}(t')\, U_1(t' - t)\, U_0(t)\,, \tag{5.3}$$

$$S_1 = \lim_{t' \to \infty,\, t \to -\infty} U_1^{-1}(t')\, U(t' - t)\, U_1(t)\,, \tag{5.3'}$$

are existing, then

$$S = \lim_{t' \to \infty,\, t \to -\infty} U_0^{-1}(t')\, U(t' - t)\, U_0(t)\,, \tag{5.3''}$$

is existing too.

This may be proved remarking that if S_0 does exist it exists also

$$\Sigma_0 = \lim_{t \to \infty} U_0^{-1}(t)\, U_1(t)\,,$$

$$\Sigma_0' = \lim_{t \to -\infty} U_1(t)\, U_0(t)\,.$$

In fact, as we have already remarked, a necessary and sufficient condition for the existence of S_0 is that:

$$\lim_{t' > t,\; t \to \infty} U_0^{-1}(t')\, U_1(t' - t)\, U_0(t) = 1\,,$$

(and the similar condition for $t' \to -\infty$).

But the preceding relation is also a necessary and sufficient condition for the existence of Σ_0. Indeed it is sufficient to remark that:

$$U_0^{-1}(t')\, U_1(t') - U_0^{-1}(t')\, U_1(t' - t)\, U_1(t) = U_0^{-1}(t')\, U_1(t' - t)\, U_0^{-1}(t)\, U_0(t)\, U(t)\,.$$

Then of course necessary and sufficient conditions for the existence of S and S_1 are respectively the existence of the limits:

$$\Sigma = \lim_{t \to \infty} U_0^{-1}(t)\, U(t)\,, \qquad \text{(and } \Sigma'\text{)},$$

$$\Sigma_1 = \lim_{t \to \infty} U_1^{-1}(t)\, U(t)\,, \qquad \text{(and } \Sigma_1'\text{)},$$

but

$$U_0^{-1}(t)\, U(t) = U_0^{-1}(t)\, U_1(t)\, U_1^{-1}(t)\, U(t)\,.$$

This result, remembering that all the operators under consideration are bounded, is sufficient to prove the existence of S. In our case we identify H_1 with (5.2) and then we may suppose safely the existence of the operator S_0. The only thing which we have to do is therefore to construct the operator S_1 which has the meaning of an S-matrix defined taking as a free Hamiltonian not H_0 but $H_0 + H_1$.

For this purpose, we try to correct the interaction H_2 by a counter term H_c in such a way that the scalar product of any eigenstate of positive norm of $H_0 + H_1$ by any eigenstate of negative norm of $H + H_c$ be equal to 0. Let us suppose that we have determined some H_c satisfying this condition. Then we may construct the correspondence C in the following way. First we put in 1-1 correspondence the eigenstate of positive norm $\beta'_i)$ of the Hamiltonian H with the eigenstate of positive norm $\beta'_{ic})$ of the Hamiltonian $H + H_c$. Then we put in correspondence $V_g)$(physical eigenstate of one V particle corresponding to g) with $V_{g'})$ and furthermore, to any state vector which behaves asymptotically (for a very far past or future) like

$$\psi) = \sum \varphi_i a_i^* V) + \sum \varphi_{lk} a_k^* a_l^* N) \tag{5.4}$$

(where φ_i and φ_{lk} define wave packets which in the limiting case of very far past or future and $\Omega \to \infty$ do not have any spatial superposition with V) (*) we put in correspondence

$$\psi') = \sum \varphi_i a_i^* V_g) + \sum \varphi_{lk} a_k^* a_l^* N) \,. \tag{5.4'}$$

Finally we express $\psi')$ by means of the eigenstate of positive norm of $H + H_c$. This is actually possible given our hypothesis about H_c and given the fact that $\psi')$ may be expressed by means of the eigenstate of positive norm of the Hamiltonian $H_0 + H_1$. So if

$$\psi') = \sum \psi_i \beta'_{ci}) \,, \tag{5.4''}$$

to $\psi)$ in $\mathcal{H}_2$ will correspond

$$\psi'') = \sum \psi_i \beta'_i) \,. \tag{5.4'''}$$

The correspondence C is so defined in such a way to conserve scalar products as required because both sets $\beta'_i)$ and $\beta'_{ic})$ which define the correspondence are orthogonal sets, and because of properties of asymptotical localized states (cf. Appendix).

(*) Cfr. Appendix.

Now we have simply to indicate how to determine H_c and to show that the correspondence C which we have constructed in this manner satisfies the condition of allowing the definition of the S-matrix S_1.

We shall indicate with β_i) the eigenstate of positive norm of H_0+H_1, with γ_i) the eigenstate of negative norm of H_0+H_1, and with γ'_{ci}) the eigenstate of negative norm of $H+H_c$. The condition which we have imposed is

$$(\beta_i|\gamma'_{cl}) = 0 \tag{5.5}$$

for all i and l.

We shall prove, now, that the necessary and sufficient condition for the relation (5.5) is

$$(\beta'_{ci}|H_2+H_c|\gamma'_{cl}) = 0\,. \tag{5.5'}$$

Let E_n and E'_n be the eigenvalue of H_0+H_1 and $H+H_c$ respectively. Then:

$$\begin{aligned} E_n\beta_n) = (H_0+H_1)\beta_n = (H+H_c-(H_2+H_c)\beta_n) = \\ = \sum E'_m\cdot\beta'_{cm})(\beta'_{cm}|\beta_n) + \sum E'_{l'}\gamma'_{cl})(\gamma'_{cl}|\beta_n) - \\ - \sum (H_2+H_c)\cdot\beta'_{cm})(\beta'_{cm}|\beta_n) - \sum_1^l (H_2+H_c)\gamma'_{cl})(\gamma'_{cl}|\beta_n)\,. \end{aligned}$$

It follows:

$$\begin{aligned} E_n(\gamma'_{cl}|\beta_n) = E'_l(\gamma'_{cl}|\beta_n) - \sum_m (\gamma'_{cl}|H_2+H_c|\beta'_{cm})(\beta'_{cm}|\beta_n) - \\ - \sum_p \gamma'_{cl}|H_2+H_c|\gamma'_{cp})(\gamma'_{cp}|\beta_n)\,, \end{aligned}$$

and if (5.5') is satisfied,

$$(E_n-E'_l)(\gamma'_{cl}|\beta_n) = -\sum_p (\gamma'_{cl}|H_2+H_c|\gamma'_{cp})(\gamma'_{cp}|\beta_n)\,. \tag{5.6}$$

In an analogous way we get:

$$(E_n-E'_n)(\beta'_{lm}|\beta_n) = -\sum_q (\beta'_{cm}|H_2+H_c|\beta'_{cq})(\beta'_{cq}|\beta_n) \tag{5.6'}$$

and a similar equation for $(\gamma'_{cl}|\gamma_n)$.

It follows that if (5.5') is satisfied, first the equations for $(\beta'_{cm}|\beta_n)$, $(\gamma'_{cl}|\beta_n)$ and $(\gamma'_{cl}|\gamma_n)$ are not only independent but they are not connected. Second, the Eq. (5.6) has the solution 0 as we want. The condition (5.5') is therefore sufficient for (5.5); it is clear that it will be also necessary if we ask that (5.5) holds identically with g'.

Now we like to make a remark. In general when there is crossing of levels E'_n, E_m due to some interaction the perturbation treatment of the interaction

will not be allowed. The reason of this fact is due to the espression $1/(E'_n - E_m)$ which appears in the perturbation expansion (of the « Wigner type ») and this expression gives rise to singularities due to the crossing. However, this will not be any more the case if all the terms of the expansion containing a denominator with crossing are identically equal to 0 because of the vanishing of the related matrix elements of the interaction. Now we are just in a similar situation, if the condition (5.5′) holds. In fact, we shall have crossing of energy levels due to the fact that the energy of the ghost state is increasing when we increase the coupling constant. Therefore the eigenvalue of some of the states of negative norm will cross the eigenvalue of some other state of positive norm when we give a variation to g'. However, following our preceding remark, this fact will not produce any harm in a perturbation expansion of the transformation function $(\beta'_{cm}|\beta_n)$. In fact, this transformation function, owing to the condition (5.5′) will satisfy the equation

$$(5.6'') \qquad (\beta'_{cm}|\beta_n) = -\frac{(\beta'_{cm}|H_2 + H_c|\beta_n)}{E_n - E'_m}(\beta'_{cn}|\beta_n) - \sum_{p\neq n}{}' \frac{(\beta'_{cm}|H_2 + H_c|\beta'_{cp})}{E_n - E'_m}(\beta'_{cp}|\beta_n),$$

and never crossing denominators will appear in the expansion obtained by iteration. One has to notice that we always renormalize the mass of the V physical state in such a way that it is independent of the value of g'. Of course, as the transformation between β_m) and β'_{cn}) is a unitary transformation, the possibility of the expansion obtained by iteration of (5.6″) is sufficient to grant us that it is possible to pass from H_0+H_1 to $H+H_c$ by means of a perturbation expansion.

We can then use this circumstance first in order to determine H_c by means of an expansion in powers of Δg. We shall write

$$(5.7) \qquad \begin{cases} H_c = \Delta g W\,, \\ W = W_0 + \Delta g W_1 + \dots\,, \\ H_2 = \Delta g H'\,. \end{cases}$$

We have now to determine by successive steps W_0, W_1 etc. Let us determine W_0; we get:

$$(\beta'_{cn}|H' + W|\gamma'_{ci}) = (\beta_n|H'\gamma_i) + (\beta_n|W_0\gamma_i) + \\ + \Delta g\left\{(\beta_n|W_1\gamma_i) + \left(\sum \delta_r \frac{(\beta_n|H' + W_0|\delta_r)}{E_{\beta_n} - E_{\delta_r}}\middle|(W + H')k_i\right) + \right. \\ \left. + \left(\beta_n|H' + W|\delta_r \frac{(\delta_r|H' + W|\gamma_i)}{E_{\gamma_c} - E_{\delta_r}}\right)\right\},$$

where δ_r) is any β) or γ) state. Therefore we may put:

$$(5.7') \qquad \begin{cases} (\beta_r|W_0|\beta_n) = 0\,; \qquad (\gamma_l|W_0)|\gamma) = 0 \\ (\beta_r|W_0|\gamma_s) = -(\beta|H'|\gamma_s) \end{cases}$$

and W_0 will be completely determined by (5.7') (*).

Proceeding in this way, we may determine step by step the operators W_1, W_2, etc. For instance one gets that W_1 may be put equal to 0.

It remains now to control whether this procedure allows us to define the S-matrix S_1 For this purpose we recall the proof of the existence of the S-matrix given in the previous Section in the case of the « double-band model ». One may easily recognize that we have made use essentially only of the fact that the transformation function between the eigenstate of KUK^{-1} and the eigenstate of U_0 in the limit of $\Omega \to \infty$ could be put under the form

$$(5.8) \qquad \beta_{m,k} = B_k \delta(E_m - E_k) + \frac{\overline{\beta}_{m,k}}{E_m - E_k - i\varepsilon},$$

there $\overline{\beta}_{m,k}$ may be prolongated in an analytical function limited in a neighbourhood of the real axis interval on which $\overline{\beta}_{m,k}$ is directly defined.

Now, if we keep in mind our condition about the self energy of the V particle (which has to remain unchanged), we see that if we use a perturbation expansion, the transformation function $(\beta_m|\beta_{cn})$ between the states β_m) and β'_{cn}) may again be put under the form (5.8).

If we consider only the first order of the perturbation expansion, then the functions $\beta_{m,n}$ are merely the matrix elements of the interaction H_2 between the states β_m) and β_n). Now, the decisive point to notice is that our procedure of determining the counter terms H_c merely suppresses the matrix elements of the interaction between β_m) and γ'_{cl}) and has no influence whatsoever on the matrix elements between β_m) and β_n). This procedure, therefore, does not modify in no way the possibility of prolongating the function $\overline{\beta}_{m,n}$ in a function having the wanted analytical properties. This is essentially due to the fact that the eigenstate with positive and the eigenstate with negative norm of the Hamiltonian H_0+H_1 do not go the one into the other in any continuous way. It may be further remarked that as our procedure may be obviously iterated the proof reached for first order perturbation expansion may

(*) There may be a correction in the self-energy term of the V-particle which has no importance for our purpose.

be immediately extended, at least so far as the transformation function may be prolongated as analytical function of the coupling constant g without going out of the real axis.

The same remark about the behaviour of the wave packets applies in this case as in the case of the « double-band » model.

Before closing this Section, we would like to add a few remarks. The first is concerning the fact that we have to make an essential use of a kind of cut-off in the form factor f_i. This fact, of course, would be rather annoying in a relativistic theory, but has no importance in the present case. Anyhow, the relevant remark is that we may construct a theory even with values of the coupling constant greater than the critical values if we take a cut-off. But as the cut-off may be as large as we like, this fact does not prevent to think of the possibility of the theory without a cut-off and with a coupling constant different from 0 as a limit of the case with a finite cut-off when the cut-off is going to infinity. The second remark is about the possibility of extension to a relativistic case. This possibility is very far from being obvious. However, we would like to point out that the rule of putting $=0$ *all* the matrix elements of the interaction between states of positive norm and states of negative norm seems in a certain way not too dangerous with respect to possibilities of a relativistic extension, if, granted the existence of eigenstates of the energy of positive norm and the eigenstate of the energy of negative norm, even in the relativistic case, one thinks that this subdivision has to be relativistic invariant.

Third, we notice that the « asymptotically localized states » do not form a complete set even in $\mathcal{H}_2$ and consequently the proposed procedure could give rise to new « bound states ». This, however, does not seem to be the case in a simple Lee model.

6. – Conclusion.

We have shown that for the Lee model it is possible, with an appropriate redefinition of the in-going and out-going states, to construct a unitary S-matrix, even in the case in which the characteristic equation for the eigenvalues of the energy in the sector V, θN has only real distinct roots. However, against our procedure the objection may be obviously raised that it is not an unique prescription and therefore it is arbitrary. This is without any doubt so; but, we think that we have to expect this arbitrariness when we introduce the indefinite metrics. In fact, the introduction of the indefinite metrics does not allow any more to maintain the usual interpretation of states of quantum mechanics *and* to have a localized description of the system. Therefore, es-

sentially less detailed information about systems is possible than is in quantum mechanics with a positive definite metrics.

Consequently, in order to have a definite theory we have to establish which are the observables in the case of the indefinite metrics and in which manner these observables may be used in order to define the states. Without answering this question, it is obvious that the interpretation of the theory does remain arbitrary. However, it is sufficient to fix already some one of the observables to reduce greatly the arbitrariness of the theory.

In our example this is evident; in general, we had to establish some homomorphic correspondence between pseudo Hilbert space $\mathcal{H}_1$ and Hilbert space $\mathcal{H}_2$. Having fixed the energy as one of the observables already reduces enormously the possible homomorphism. As we have remarked in the introduction, the possible homomorphisms are reduced, indeed, to isomorphism between the subspace of $\mathcal{H}_1$, formed by the closure of the linear manifold of the eigenstate of the energy of positive norm, and $\mathcal{H}_2$. However, the energy only and the requirement that the asymptotical motion of the wave packets has to be described in exactly a similar manner as the motion of free wave packets are not sufficient to fix this isomorphism, and for that purpose it would be necessary to specify other observables of the system.

It is quite obvious that a further research would be hardly worth-while for answering this question for the Lee model. This model is too far from reality, and therefore as far as the Lee model is concerned, we have probably to be satisfied with the results which we have reached in the present investigation.

In conclusion, we have shown the possibility of an approach to the problem of the physical interpretation of a quantum theory with an indefinite metric, which is perhaps in some aspect new and which, as far as we know, is not much worst than the previous approach.

Whether it is any better it is a question which presently I cannot answer.

Appendix

We would like, in this appendix, to indicate how the notion of the localization of the 0-particle may be defined in the Lee model. Let us start with a trivial case in which we have not any « heavy » particles present, and let us consider the sector in which we have only one 0-particle. Then we consider the operator

$$H_f = \int f^2(\boldsymbol{x})[\pi_0^2 + (\operatorname{grad} \psi_0)^2 + \mu^2\psi_0^2]\,\mathrm{d}\boldsymbol{x}\,. \tag{A.1}$$

The function f^2 defines a dominion in space (the « region f^2 ») in which it is different from zero.

The operators ψ_θ and π_θ are the operators of the θ field and its canonical conjugate operator. It is easy to prove that the operator (A.1) first has no negative eigenvalue (for this purpose it is sufficient to show that the expectation value of H_f is always positive or 0 for any state belonging to the sector $1\,\theta$). Second: supposing that f^2 is different from zero only in a limited domain in the limit $\Omega \to \infty$ it is possible to construct states belonging to the sector $1\,\theta$ for which the expectation value for the operator H_f is less than any given quantity.

This means that in this case the operator H_f has eigenvalues which are smaller than any prefixed positive quantity.

In the case in which the volume Ω where we quantize the system is not actually infinite if the domain in which $f^2 \neq 0$ is much smaller than Ω, it is possible to show that one can always construct states in such a way that the expectation value of the operator H_f is $0(\exp[-\Omega^{\frac{1}{3}}\mu])$.

Given then some f^2 for which it does exist some eigenvalue of the operator H_f less than ε we may consider the subspace of all the state functions belonging to the sector $1\,\theta$ and which are normal to the manifold of the eigenstate of the operator H_f having an eigenvalue $\leqslant \varepsilon$. This subspace will be called the subspace of states in which the θ-particle is localized out of the region defined by f^2 with the precision ε.

Of course, this notion may be extended even to a case in which the function f^2 is equal to 0 only in a limited domain. If the domain contains a sphere of radius R the precision may be of order $\exp[-R\mu]$. Having defined the notion of localization with a precision ε out of a region defined by a function f^2, we have at the same time defined the notion of the localization inside the region defined by $1 - f^2$.

Coming back to the case in which f^2 is different from 0 only in a limited domain, R_{f^2}, we may define the notion of an « asymptotical localization » out of R_{f^2}, if the precision of the localization out of R_{f^2} is going to zero.

Of course the notion of asymptotical localization may be extended immediately to the case in which f^2 is not exactly 0 out of a finite domain, but is going to 0 with a sufficient rapidity with increasing distance from a fixed point.

Now we may extend these definitions to the case in which we have a θ-particle in presence of a « heavy » particle. The case in which the « heavy » particle is an N particle is trivial. We shall consider therefore briefly the case of a θ-particle in presence of a physical V-particle. Let us consider the vector:

$$\psi) = \sum_k \chi_k a_k^* V) \,, \tag{A.2}$$

where

$$V) = (\psi_V + N \sum \Phi_i a_i^*) \cdot 0) \,, \tag{A.2'}$$

is the state in which we have a physical V-particle.

Now we notice that the function:

$$f_{\mathrm{V}}^2 = |\sum \Phi_l \exp[i\boldsymbol{k}_l \cdot \boldsymbol{x}]|^2 , \tag{A.3}$$

is going to 0 exponentially when $|\boldsymbol{x}|$ is going to infinite and we may therefore define the asymptotical localized state for the 1 θ sector with respect to the function f_{V}^2 following the definition which we have given. It is immediately verified that if the vector $\chi \cdot 0)$, where

$$\chi = \sum \chi_k a_k^* , \tag{A.4}$$

is localized out of $R_{f_{\mathrm{V}}^2}$ with the precision ε then

$$(0|[(\psi_{\mathrm{V}} + N \sum \Phi_l a_l^*), \chi]_- 0) = 0(\varepsilon) . \tag{A.5}$$

It follows also that if $\chi_1 \cdot 0)$ and $\chi_2 \cdot 0)$ are two orthogonal states localized out of R_{f^2} with the precision ε,

$$(V\chi_1 | \chi_2 V) = 0(\varepsilon) . \tag{A.6}$$

The relations (A.5) and (A.6) allow us to talk in the case of asymptotical localization out of $R_{f_{\mathrm{V}}^2}$ of states representing a particle θ in the state $\chi \cdot 0)$ and one physical particle V without any ambiguity.

This is not always the case; for instance this is not the case for the states $a_k^* V)$. We cannot say that this state does represent a particle θ of momentum k and a physical V-particle. In fact, the operator a_k^* does not commute with the operator $\psi_{\mathrm{V}} + N \sum \Phi_l a_l^*$. It is true that the commutator is of the order $1/\Omega^{\frac{1}{2}}$ where Ω is as usual the volume of integration, and therefore it goes to 0 when the volume is going to infinity. However, it does not go to zero with a sufficient rapidity. This may be easily seen in the following manner: Let us consider the set of states

$$\varphi_k) = a_k^* V) , \tag{A.7}$$

and let us suppose that $\psi_1)$ and $\psi_2)$ are any two states which may be represented by the following expression

$$\psi_1) = \sum \psi_k^{(1)} \varphi_k) , \qquad \psi_2) = \sum \psi_k^{(2)} \varphi_k) . \tag{A.7'}$$

If we could consider even in the limit of $\Omega \to \infty$ the set (A.7) as an orthogonal set, then it should be $(\psi_1 | \psi_2) = \sum \psi_k^{(1)*} \psi_k^{(2)}$. Instead we have exactly:

$$(\psi_1 | \psi_2) = \sum \psi_k^{(1)*} \psi_k^{(2)} + \frac{g^2}{2\Omega} \sum_i \frac{f_i \psi_{k_i}^{(1)*}}{\omega_i} \sum_i \frac{f_i \psi_{k_i}^{(2)}}{\omega_i} . \tag{A.8}$$

Now $\psi_k \sim 1/\Omega^{\frac{1}{2}}$ and therefore, remembering that $\sum f_i \sim \Omega$, it follows that $1/\Omega^{\frac{1}{2}} \sum f_i \psi_{ki}/\omega_i$ is in general different from 0, even in the case of $\Omega \to \infty$. Therefore the states $a^*_{k_1} V)$ and $a^*_{k_2} V)$ cannot be considered orthogonal, not even in the limit $\Omega \to \infty$ and consequently it cannot be said with a clear meaning that they do represent a θ-particle with a momentum k_1, and respectively k_2, together with a physical V-particle. The case of asymptotically localized packet is different because the precision ε may go to zero *exponentially* with $\Omega^{\frac{1}{2}} \to \infty$.

We would like now to make some remark about the procedure followed in Sects. **4** and **5**. Let us consider, for instance, the procedure of Sect. **5**.

There we were considering two different values of the coupling constant g and g' and the corresponding states of the physical V-particle V_g) and $V_{g'}$). It is essential to remark that if $\chi \cdot 0)$ is asymptotically localized out of $R_{f^{\frac{1}{2}}_{V_g}}$ it is also asymptotically localized out of $R_{f^{\frac{1}{2}}_{V_{g'}}}$. There is not therefore any difficulty whatsoever from a kinematical point of view when one represents a physical situation in which there is a physical V-particle and a θ-particle in the asymptotical localized state $\chi \cdot 0)$ by means of $\chi \cdot V_n)$ even in the case in which the coupling constant has a value g', following the prescription of Sect. **5**.

We would like to add a last remark. Let us consider a « one band » model in which, however, the form factor is given by a Gaussian. It is then easy to verify that a Lee model of this kind is not worst, about « locality », than the usual one. Now, it is easy to find that the standing waves in the sector θV corresponding to the case of a θ-particle in presence of a V-particle are of the form

$$\text{(A.9)} \qquad \psi^{(m)}) = \sum_l \psi_l^{(m)} a_l^* \left[NII_{\mathrm{V}} 00) + \frac{g}{\sqrt{2\Omega}} \sum_r \frac{f_r}{\omega_r} \cdot I_{\mathrm{N}} I_r) \right] + \\ + \frac{g}{\sqrt{2\Omega}} \sum_{l,r} \psi_l^{(m)} \frac{z_m - \omega_l}{\omega_r^2} f_r \cdot I_{\mathrm{N}} I_l I_r) \,,$$

with

$$\text{(A.9')} \qquad \psi_l^{(m)} = c_m f_l \left[\frac{1}{(m - l + \eta_m)} + c'_m \right].$$

It has to be noted that this is the form in the case in which $g < g_c$. Now in an ordinary case of a system having forces with a finite range, one should expect that in case of two particles, the standing waves in the limiting case of $\Omega \to \infty$ should be simply a product of standing waves of the first particle by standing waves of the second particle. This is not the case with our model because the last term of the second member of (A.9) is not going to 0 when $\Omega \to \infty$. This is a rather strange behaviour and it may originate the suspect that even when $g < g_c$ a θ-particle is never dynamically independent from a V-particle, also when it is localized at a very far distance from the V-particle. This, however, is not the case (when $g < g_c$) and this may be verified applying to the standing wave represented by Eq. (A.9) any operator χ^* where $\chi \cdot 0)$ represents an asymptotically localized θ-particle state. It is then possible to verify that the last term of the second member of Eq. (A.9) does not give

any contribution. That means that when the θ-particle is really localized at a very far distance of the V-particle, the V-particle is left unchanged. Therel fore, one may be fairly sure that the trouble which arises in the Lee modewhen $g > g_c$ depends really from the negative metric and not from any peculiarity of the Lee model itself.

RIASSUNTO

Nel presente lavoro si discute di nuovo il modello di Lee nel caso già considerato da PAULI e KÄLLÉN, cioè nel caso in cui le radici dell'equazione caratteristica dell'energia sono tutte reali e distinte. Si mostra che è possibile ridefinendo gli stati entranti e uscenti costruire una matrice S unitaria e con proprietà fisiche ragionevoli. Il procedimento suggerito è in qualche modo simile a un procedimento di rinormalizzazione ed elimina gli effetti dei fantasmi.

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On the Existence and the Unitary Property of the Scattering Operator.

S. T. Kuroda

Department of Physics, University of Tokyo - Tokyo

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Summary. — The mathematical character of the scattering theory is investigated according to the « time-dependent » formulation with special emphasis on the wave and the scattering operators. The main contents are: 1) some general properties of these operators are summarized; 2) the existence of the wave operator is proved under rather mild conditions on the potential; and 3) the existence and the unitary property of the scattering operator are proved under certain additional conditions. The arguments are mathematically rigorous.

1. – Introduction.

The properties of quantum-mechanical scattering systems are most conveniently described by the so-called scattering operator or the S-matrix. It is an operator which assigns the state in the distant future to the state in the remote past and is usually expected to be a unitary operator in the basic Hilbert space $\mathfrak{H}$. From the mathematical point of view, however, the arguments given in most of the physical literature are of rather formal character and it seems that, even in the scattering problem of elementary wave mechanics, mathematically rigorous treatment has not been carried out sufficiently in spite of the importance of the scattering operator. The purpose of the present paper is to study the mathematical aspects of the theory and to prove rigorously the existence and the unitary property of the scattering operator in simple problems of quantum mechanics.

Recently J. M. Jauch [1] has stressed the importance of the rigorous treatment of the problem and has given a mathematically suitable formulation of

[1] J. M. Jauch: *Helv. Phys. Acta.*, **31**, 127 (1958).

the theory of the scattering operator with detailed physical interpretations. Since our research, which was done independently of JAUCH's work, is also based on an analogous formulation, we shall state the problem along the line given by JAUCH. Throughout the present work we shall be concerned with the scattering system in which the so-called unperturbed Hamiltonian H_0 is clearly prescribed. Then the existence and the properties of the scattering operator are closely connected with the asymptotic properties of the family of unitary operators $U_t = \exp[itH_1]\exp[-itH_0]$ (*), where H_1 denotes the total Hamiltonian of the system. For the moment we assume that, roughly speaking, H_0 has no point eigenvalues. Then the strong limits $W_\pm$ of U_t for $t \to \pm\infty$ are called *wave operators*, if they exist. The *scattering operator* S is then defined by $S = W_+^* W_-$.

According to JAUCH [1], the fundamental problems of the theory are now as follows.

Problem (I): *To prove the existence of the wave operators* $W_\pm$.

Problem (II): *To prove that the scattering operator* S *is unitary.*

Problem (II) has been shown to be equivalent to the following

Problem (II'): *To prove that* $W_+\mathfrak{H}$ *coincides with* $W_-\mathfrak{H}$.

(See also Corollary to Theorem 3.1.) JAUCH [1] examined some properties of the wave and the scattering operators under the assumption that the problems (I) and (II) have been proved. However, it seems no less important to solve the problems (I) and (II) for concrete physical systems. Heretofore this has been done by J. M. COOK [2] and J. M. JAUCH and I. I. ZINNES [3]. They considered the system with the Hamiltonian given by the differential operator $H_1 = -\Delta + V(x)\cdot$ in the 3-dimensional configuration space with the co-ordinates $x = (x_1, x_2, x_3)$, where $V(x)$ is a given potential. COOK [2] solved the problem (I) under the assumption that $|V(x)|^2$ is integrable in the whole configuration space. JAUCH and ZINNES [3] assumed that $V(x) = |x|^{-\beta}$ $1 < \beta < \frac{3}{2}$ and solved the problem (I). But these authors do not give any results on the problem (II).

On the other hand, the theory of the scattering operator is closely related to the mathematical problem concerning the *stability of the continuous spectra* of self-adjoint operators. This problem has been studied by K. O. FRIED-

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(*) Here we must remark on the deviation of our notation from that adopted by JAUCH. In [1] JAUCH uses the notation $U_t = \exp[-itH_0]$ and the present U_t is denoted in [1] by $V_+^* U_t$, $V_t = \exp[-itH_1]$. The wave operators $W_\pm$ which will appear in the following are denoted in [1] by $\Omega_\mp$.

[2] J. M. COOK: *Journ. Math. Phys.*, **36**, 82 (1957).

[3] J. M. JAUCH and I. I. ZINNES: *Nuovo Cimento*, **11**, 553 (1959).

RICHS (4), T. KATO (5,6) and others (7,8) from various standpoints and by different methods. According to KATO, it is convenient for the proof of the problem (II) to generalize the notion of the wave operator to some extent so as to include the case in which H_0 has point eigenvalues. The wave operator in the generalized sense will be called the *generalized wave operator*. The exact definition and the detailed explanation of the reason of introducing it will be given in Section **3**. NOW, KATO (5,6) solved the problems (I) and (II) in the generalized sense, by assuming that $H_1 = H_0 + V$, where V satisfies a condition which implies a *mathematically* simple structure of V. Unfortunately, this theorem is not applicable to concrete physical problems because the assumption on V is too restrictive. However, we can weaken the assumption in this theorem by a sort of limiting procedure and in this generalized form it can be used to solve the problem (II) for physically interesting cases. This generalization of Kato's theorem is stated below as Lemmas 5.1 and 5.2 without proof; their proof, which is rather long and of purely mathematical character, will be given in a separate paper (KURODA (9)).

The contents of the present paper are as follows. In Section **3** we give the definition of the generalized wave and scattering operators and examine some of their important properties. This section has a character of a review to some extent, for its contents have been partly stated by several authors here and there. However, it seems worth-while to summarize them in the possibly most general form.

In Section **4** we shall treat the physical system described by the Hamiltonian: $H_1 = -\Delta + V(x)$ and solve the problem (I) under a weaker condition on $V(x)$ than that imposed by COOK (2) or JAUCH and ZINNES (3) (Theorem 4.1). We shall also solve the problem (II) in some special cases (Corollary to Theorem 4.1.). Our result is not restricted to the case in which the space dimension is equal to 3 and the application to many particle problems will also be mentioned (Theorem 4.2). We had obtained these results independently of that of COOK (2) and JAUCH and ZINNES (3) by a method somewhat analogous to that of (3). However, being suggested by the method used in (3), we have been able to simplify the proof of Theorem 4.1 to a considerable extent (*).

In Section **5** we shall state the lemmas mentioned above and apply them

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(4) K. O. FRIEDRICHS: *Comm. Pure Appl. Math.*, **1**, 364 (1948).
(5) T. KATO: *Journ. Math. Soc. Japan*, **9**, 239 (1957).
(6) T. KATO: *Proc. Japan Acad.*, **33**, 260 (1957).
(7) N. ARONSZAJN: *Amer. Journ. Math.*, **79**, 597 (1957).
(8) M. ROSENBLUM: *Pacif. Journ. Math.*, **7**, 997 (1957).
(9) S. T. KURODA: *Perturbation of continuous spectra by unbounded operators*, to be published in *Journ. Math. Soc. Japan*.
(*) The writer is indebted to Professor J. M. JAUCH for sending him a preprint of (3) before formal publication.

to the same problem as in Section **4**, solving the problem (II) together with (I) under somewhat more restrictive assumptions than in Section **4**.

Finally in Section **6** some remarks will be given on the nature of the spectrum of the partial differential operator $-\Delta+V(x)\cdot$.

The arguments in the present paper are based on the operator theory of Hilbert space. A brief review of the theory is given in JAUCH ([1]). We give in Section **2** a brief account of those results of the theory of Hilbert space which are necessary for our purpose and not contained in JAUCH's article.

2. – Preliminaries.

2·1. *Basic terminologies and notations.* – We denote the underlying (not necessarily separable) Hilbert space by $\mathfrak{H}$ (*). The inner product and the norm in $\mathfrak{H}$ are denoted by (u, v) and $\|u\| = (u, u)^{\frac{1}{2}}$ respectively. The domain of an operator A in $\mathfrak{H}$ is denoted by $\mathfrak{D}(A)$. In this paper all the operators are assumed to be linear and have the domain dense in $\mathfrak{H}$ unless otherwise stated. The adjoint of an operator A is denoted by A^*. An operator B is said to be an extension of A and denoted by $B \supset A$, if $\mathfrak{D}(B) \supset \mathfrak{D}(A)$ and $Bu = Au$ for every $u \in \mathfrak{D}(A)$. We freely avail ourselves of the notions such as *bounded operators*, *projections*, *unitary operators*, *symmetric operators* and *self-adjoint operators* (see *e.g.* JAUCH ([1]) or STONE ([10]), Chapt. II). According to the spectral theorem for a self-adjoint operator, there corresponds to each self-adjoint operator H a family of projections $\{E(\lambda)\}$ (called the *resolution of the identity* corresponding to H) such that $H = \int \lambda \mathrm{d}E(\lambda)$ (see *e.g.* ([1]) or ([10]), Chapt. V). By means of this theorem we can define functions of a self-adjoint operator H (see *e.g.* ([1]) or ([10]), Chapt. VI). In particular, the unitary operator $\exp[itH] = \int \exp[it\lambda] \mathrm{d}E(\lambda)$ plays an important rôle in the sequel. We shall also use the operator $|H|^{\frac{1}{2}} = \int |\lambda|^{\frac{1}{2}} \mathrm{d}E(\lambda)$ in Sect. **5**.

The m-dimensional Euclidean space is denoted by E_m and points or position vectors of E_m are denoted by x, y, k, etc. By means of the cartesian co-ordinate system x is represented as $x = (x_1, \ldots, x_m)$. We denote by xy the inner product $x_1y_1 + \ldots + x_my_m$ of two vectors x and y. In particular, the absolute value or the length $(x^2)^{\frac{1}{2}}$ of x is denoted by $|x|$. $L^2(E_m)$ and $L^1(E_m)$ are by definition the sets of all functions $u(x)$ defined on E_m such that

(*) For details concerning the theory of Hilbert space, see, for example, STONE ([10]), RIESZ and SZ.-NAGY ([11]) and VON NEUMANN ([12]).

([10]) M. H. STONE: *Amer. Math. Soc. Coll. Publ. XV* (New York, 1932).

([11]) F. RIESZ and B. VON SZ.-NAGY: *Functional Analysis* (New York, 1955).

([12]) J. VON NEUMANN: *Mathematical Foundation of Quantum Mechanics* (Princeton, 1955).

$\int |u(x)|^2 \mathrm{d}x < \infty$ (*) and $\int |u(x)| \mathrm{d}w < \infty$, respectively. $L^2(E_m)$ is a Hilbert space with the inner product $(u, v) = \int u(x)\overline{v(x)}\,\mathrm{d}x$. We denote by $\hat{u}(k)$ the Fourier transform of the function $u(x)$:

$$\hat{u}(k) = (2\pi)^{-m/2} \int u(x) \exp[-ikx]\,\mathrm{d}x\,.$$

As is well known, when $u(x)$ belongs to $L^2(E_m)$, $\hat{u}(k)$ also belongs to $L^2(E_m)$ and the mapping $u(x) \to \hat{u}(k)$ is a unitary transformation.

2·2. *Partially isometric operators and unitary equivalence.* – Let $\mathfrak{M}$ and $\mathfrak{N}$ be two subspaces (**) having the same dimension. A bounded operator U is said to be *partially isometric* (MURRAY and VON NEUMANN ([13])) with the initial set $\mathfrak{M}$ and the final set $\mathfrak{N}$ if U transforms $\mathfrak{M}$ isometrically onto $\mathfrak{N}$ and $\mathfrak{H} \ominus \mathfrak{M}$ (**) to $\{0\}$, that is, if $U\mathfrak{M} = \mathfrak{N}$ (**), $\|Uu\| = \|u\|$ for $u \in \mathfrak{M}$ and $Uu = 0$ for $u \in \mathfrak{H} \ominus \mathfrak{M}$. This is equivalent to the condition that

$$U^*U = P\,, \qquad UU^* = Q\,, \tag{2.1}$$

where P and Q are (orthogonal) projections on $\mathfrak{M}$ and $\mathfrak{N}$ respectively. In this case U^* is also a partially isometric operator with the initial set $\mathfrak{N}$ and the final set $\mathfrak{M}$. Moreover, if V is a partially isometric operator with the initial set $\mathfrak{N}$ such that $VU = P$, then $V = U^*$. If in particular $\mathfrak{M} = \mathfrak{H}$ (*i.e.* $P = I$), U is called an *isometric operator* with the final set $\mathfrak{N}$. Furthermore, U is a unitary operator if $\mathfrak{M} = \mathfrak{N} = \mathfrak{H}$ (*i.e.* $P = Q = I$).

A subspace $\mathfrak{M}$ is said to *reduce* an operator A if $\mathfrak{M}$ and $\mathfrak{H} \ominus \mathfrak{M}$ are invariant by A. When A is self-adjoint, this is equivalent to the condition that $\mathfrak{M}$ is invariant by A. When $\mathfrak{M}$ reduces A, the restriction A' of A on $\mathfrak{M}$ can be regarded as an operator in $\mathfrak{M}$: in this case A' is called the *part* of A in $\mathfrak{M}$.

Two operators A and B are called unitarily equivalent to each other if there exists a unitary operator U such that $AU = UB$. For convenience this notion will be extended in the following way: Let $\mathfrak{M}$ and $\mathfrak{N}$ be two subspaces of $\mathfrak{H}$ reducing the operators A and B, respectively. The part A' of A in $\mathfrak{M}$ and the part B' of B in $\mathfrak{N}$ are said to be unitarily equivalent to each

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(*) The integral sign without any indication of the integral domain will be agreed to be the integral over the whole space E_m.

(**) We agree throughout the present paper that a « subspace » always means a closed subspace.

([13]) F. J. MURRAY and J. VON NEUMANN: *Ann. Math.*, **37**, 116 (1936).

(**) $\mathfrak{N} \ominus \mathfrak{M}$ denotes the subspace consisting of all u such that $(u, v) = 0$ for every $v \in \mathfrak{M}$, and $U\mathfrak{M}$ denotes the set of elements v such that $v = Uu$ for some $u \in \mathfrak{M}$.

other if there exists a partially isometric operator U with the initial set $\mathfrak{M}$ and the final set $\mathfrak{N}$ with the following properties: $u \in \mathfrak{M}$ belongs to $\mathfrak{D}(A)$ if and only if Uu belongs to $\mathfrak{D}(B)$ and $UAu = BUu$ for such an u. This is equivalent to the condition that

$$UAP = BQU\,, \tag{2.2}$$

where P and Q are as in (2.1). In such a case the operators A' and B' can be considered to have the same structure in an abstract sense.

2·3. *Absolutely continuous and singular parts of a self-adjoint operator.* – Usually the spectrum of a self-adjoint operator is divided into two parts, the point spectrum and the continuous spectrum. In the present paper, however, a different classification plays an important rôle. Since this classification does not seem so familiar as the former, we shall give a brief account of it. Let H be a self-adjoint operator with the resolution of the identity $\{E(\lambda); -\infty < \lambda < +\infty\}$. For each $u \in \mathfrak{H}$ consider the real-valued function $\varrho(\lambda; u) = (E(\lambda)u, u)$. As the spectral theorem shows, $\varrho(\lambda; u)$ is a non-negative, non-decreasing and right-continuous function of λ. Hence, $\varrho(\lambda; u)$ determines a positive measure $\mathrm{d}\varrho(X; u)$ on E_1 such that $\mathrm{d}\varrho((a, b]; u) = \varrho(b; u) - \varrho(a; u)$ (*). In general, a measure $\mu(X)$ on E_1 is said to be absolutely continuous (**) if $\mu(X) = 0$ for each Borel set X of Lebesgue measure 0, and $\mu(X)$ is said to be singular if there exists a Borel set Z of Lebesgue measure 0 such that $\mu(CZ) = 0$. (CZ denotes the complementary set of Z.) Now, a $u \in \mathfrak{H}$ is said to be *absolutely continuous* with respect to H if the measure $\mathrm{d}\varrho(X; u)$ is absolutely continuous, and u is said to be *singular* with respect to H if $\mathrm{d}\varrho(X; u)$ is singular. Let the set of all absolutely continuous elements of $\mathfrak{H}$ and the set of all singular elements of $\mathfrak{H}$ be denoted by $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Then we have the following lemma.

LEMMA 2.1. i) $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$, *that is,* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are mutually orthogonal subspaces and each* $u \in \mathfrak{H}$ *can be decomposed in a unique way as*

$$u = u_1 + u_2\,, \qquad u_1 \in \mathfrak{M}\,, \qquad u_2 \in \mathfrak{N}\,. \tag{2.3}$$

ii) $\mathfrak{M}$ *and* $\mathfrak{N}$ *reduce* H. *Any subspace* $\mathfrak{L}$ *of* $\mathfrak{H}$ *which reduces* H *is decomposed as*

$$\mathfrak{L} = (\mathfrak{L} \cap \mathfrak{M}) \oplus (\mathfrak{L} \cap \mathfrak{N})\,. \tag{2.4}$$

(*) X denotes a Borel set of E_1 and $(a, b]$ denotes the interval $a < x \leqslant b$. For the details on measures, see *e.g.* RIESZ and SZ.-NAGY [11], chapt. I-III or any text book on measures and integrations.

(**) We agree that « absolutely continuous » and « singular » are always referred to Lebesgue measure.

Proof. First we show that $\mathfrak{M}$ and $\mathfrak{N}$ are mutually orthogonal, that is, $(u, v) = 0$ for each $u \in \mathfrak{M}$ and $v \in \mathfrak{N}$. $v \in \mathfrak{N}$ implies that there exists a Borel set Z of Lebesgue measure zero such that $\mathrm{d}\varrho(CZ; v) = 0$. Hence we obtain $(u, v) = \int_Z \mathrm{d}(u, E(\lambda)v) = \int_Z \mathrm{d}(E(\lambda)u, v) = 0$ because $u \in \mathfrak{M}$ implies $\mathrm{d}\varrho(Z; u) = 0$. To prove i), it therefore suffices to prove (2.3); then it follows automatically that $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces. As is well known, a measure $\mathrm{d}\varrho(X; u)$ can be decomposed in a unique way as the sum of an absolutely continuous and a singular measure: $\mathrm{d}\varrho(X; u) = \mathrm{d}\varrho_a(X; u) + \mathrm{d}\varrho_s(X; u)$. For the singular measure $\mathrm{d}\varrho_s$ we can choose a Borel set Z of measure 0 such that $\mathrm{d}\varrho_s(CZ; u) = 0$. We then obtain

$$\text{(2.5)} \qquad \begin{cases} \mathrm{d}\varrho(X \cap CZ; u) = \mathrm{d}\varrho_a(X \cap CZ; u) = \mathrm{d}\varrho_a(X; u)\,, \\ \mathrm{d}\varrho(X \cap Z; u) \quad = \mathrm{d}\varrho_s(X \cap Z; u) \quad = \mathrm{d}\varrho_s(X; u)\,. \end{cases}$$

For each $u \in \mathfrak{H}$ we now define u_1 and u_2 as

$$\text{(2.6)} \qquad u_1 = \int_{CZ} \lambda\, \mathrm{d}E(\lambda)u\,, \qquad u_2 = \int_Z \lambda\, \mathrm{d}E(\lambda)u\,.$$

Then it is clear that $u_1 + u_2 = \int \lambda\, \mathrm{d}E(\lambda)u = u$. It follows easily from (2.5) and (2.6) that $\mathrm{d}\varrho(X; u_1) = \mathrm{d}\varrho(X \cap CZ; u) = \mathrm{d}\varrho_a(X; u)$. This shows that the measure $\mathrm{d}\varrho(X; u_1)$ is absolutely continuous and hence $u_1 \in \mathfrak{M}_0$. $u_2 \in \mathfrak{N}$ can be proved in the same way.

ii) Let $u \in \mathfrak{M}$ and consider $E(\mu)u$ for a fixed real μ. $u \in \mathfrak{M}$ implies that the function

$$\varrho(\lambda; E(\mu)u) = (E(\lambda)E(\mu)u, E(\mu)u) = (E(\min(\lambda, \mu)u, u)$$

is absolutely continuous. Hence $E(\mu)u \in \mathfrak{M}$. Since $E(\mu)$ is self-adjoint, it follows that $\mathfrak{M}$ and $\mathfrak{N}$ reduce $E(\mu)$ for any real μ (see Section 2·2). This implies that $\mathfrak{M}$ and $\mathfrak{N}$ reduce H (see *e.g.* STONE [10], Theorem 7.15). Next let $\mathfrak{L}$ be a subspace of $\mathfrak{H}$ reducing H. Then $u \in \mathfrak{L}$ implies $E(\lambda)u \in \mathfrak{L}$. Therefore, it follows from (2.6) that $u_1 \in \mathfrak{L} \cap \mathfrak{M}$ and $u_2 \in \mathfrak{L} \cap \mathfrak{N}$. Thus (2.4) is proved.

$\mathfrak{M}$ and $\mathfrak{N}$ in the above lemma will be called the *absolutely continuous subspace* and the *singular subspace* of $\mathfrak{H}$ with respect to H, respectively. In particular, H is said to be *absolutely continuous* if $\mathfrak{M} = \mathfrak{H}$ and *singular* if $\mathfrak{N} = \mathfrak{H}$. The spectrum of the parts of H in $\mathfrak{M}$ and in $\mathfrak{N}$ will be called the *absolutely continuous* and the *singular parts* of the spectrum of H, respectively.

3. - Generalized wave operators and generalized scattering operators.

3·1. – Let $\mathfrak{H}$ be a (not necessarily separable) Hilbert space and let H_0 and H_1 be self-adjoint operators in $\mathfrak{H}$. Then

$$U_t = U_t(H_1, H_0) = \exp[itH_1]\exp[-itH_0], \qquad -\infty < t < +\infty, \tag{3.1}$$

form a one parameter family of unitary operators in $\mathfrak{H}$. Physically, H_0 and H_1 represent the unperturbed and the total Hamiltonian, respectively. The unitary operator U_t transforms the state at the time t into the state at the time $t = 0$ in the interaction picture. We now define the generalized wave operators and the generalized scattering operator in the following manner.

DEFINITION. *Let H_0 and H_1 be as above and let P_0 be the projection on the absolutely continuous subspace $\mathfrak{M}_0$ of $\mathfrak{H}$ with respect to H_0* (see Section 2·3). *Then the* generalized wave operators $W_\pm(H_1, H_0)$ *are defined by*

$$W_+ = W_+(H_1, H_0) = s - \lim_{t \to +\infty} U_t(H_1, H_0)P_0\,, \tag{3.2}_+$$

$$W_- = W_-(H_1, H_0) = s - \lim_{t \to -\infty} U_t(H_1, H_0)P_0\,, \tag{3.2}_-$$

whenever the limit on the right side exists (*). *When W_+ and W_- both exist, the* generalized scattering operator *is defined by*

$$S = W_+^* W_- = W_+(H_1, H_0)^* W_-(H_1, H_0)\,. \tag{3.3}$$

In case H_0 is absolutely continuous, *i.e.* $P_0 = I$, the above definition coincides with the usual one for the wave and scattering operators. Since in most of the physical problems H_0 is absolutely continuous, the above generalization may seem to be merely of mathematical interest. But this is not the case; its importance in the whole theory will be seen later (see the argument before Theorem 3.3).

3·2. – We now examine some general properties of these operators, assuming their existence.

THEOREM 3.1. *Let H_0 and H_1 be self-adjoint operators with the resolutions of the identity $\{E_0(\lambda)\}$ and $\{E_1(\lambda)\}$, let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be absolutely continuous subspaces of $\mathfrak{H}$ with respect to H_0 and H_1, and let P_0 and P_1 be projections on $\mathfrak{M}_0$ and $\mathfrak{M}_1$, respectively. If $W_+ = W_+(H_1, H_0)$ exists, we have:*

(*) s – lim means the strong limit of bounded operators in $\mathfrak{H}$. Instead of saying that the strong limit in $(3.2)_\pm$ exists, we will merely say that $W_\pm$ exists.

i) *W_+ is a partially isometric operator with the initial set $\mathfrak{M}_0$ and the final set contained in $\mathfrak{M}_1$, that is,*

$$W_+^* W_+ = P_0 , \qquad W_+ \mathfrak{H} \subset \mathfrak{M}_1 . \tag{3.4}$$

ii) *W_+ satisfies the relations*

$$\exp[itH_1] W_+ = W_+ \exp[itH_0] , \qquad -\infty < t < +\infty , \tag{3.5}$$

$$E_1(\lambda) W_+ = W_+ E_0(\lambda) , \qquad -\infty < \lambda < +\infty . \tag{3.6}$$

$W_+\mathfrak{H}$ reduces H_1 and the part of H_1 in $W_+\mathfrak{H}$ is unitarily equivalent to the absolutely continuous part of H_0:

$$H_1 P_1 W_+ = W_+ H_0 P_0 , \qquad W_+^* H_1 P_1 = H_0 P_0 W_+^* . \tag{3.7}$$

If in particular $W_+\mathfrak{H} = \mathfrak{M}_1$, the absolutely continuous part of H_1 is unitarily equivalent to that of H_0.

iii) *In order that the relation $W_+\mathfrak{H} = \mathfrak{M}_1$ holds, it is necessary and sufficient that $W_+(H_0, H_1)$ also exists. In this case we have*

$$W_+(H_1, H_0)^* = W_+(H_0, H_1) . \tag{3.8}$$

The same assertions hold true for W_- in place of W_+.

COROLLARY. *Let $W_+(H_1, H_0)$ and $W(H_1, H_0)$ exist and let the relation $W_+\mathfrak{H} = W_-\mathfrak{H}$ hold. Then the scattering operator S is a partially isometric operator with the initial set $\mathfrak{M}_0$ and the final set $\mathfrak{M}_0$. The part of S in $\mathfrak{M}_0$ is a unitary operator in $\mathfrak{M}_0$. In particular, if H_0 is absolutely continuous, then S is unitary. Furthermore, S commutes with $H_0 P_0$:*

$$S H_0 P_0 = H_0 P_0 S . \tag{3.9}$$

Remark 1. As was mentioned in Section **1**, the facts stated in Theorem 3.1 are partly stated by several authors. (3.5) was proved by JAUCH [1] for the usual wave operator. (3.7) was proved by COOK [2] in the special case with which he dealt. It must be noticed, however, that in Theorem 3.1 only the existence of W_+ or W_- is assumed and we are quite free from any assumptions on the structure of $H_1 - H_0$ (*). This is by no means a trivial remark, because we can even give an example in which $W_\pm$ exist but

(*) By definition $\mathfrak{D}(H_1 - H_0) = \mathfrak{D}(H_1) \cap \mathfrak{D}(H_0)$ and $(H_1 - H_0)u = H_1 u - H_0 u$ for every $u \in \mathfrak{D}(H_1 - H_0)$.

$\mathfrak{D}(H_0)$ and $\mathfrak{D}(H_1)$ have a single common element 0, though the corresponding physical problem is not known (see (9)).

Remark 2. ii) and iii) of Theorem 3.1. give an important necessary condition which the operators H_0 and H_1 must satisfy in order that the wave operators exist. Namely, if $W_+(H_1, H_0)$ should exist, the absolutely continuous part of the spectrum of H_1 must *contain* that of H_0 as a subset and, if both $W_-(H_1, H_0)$ and $W_+(H_0, H_1)$ should exist, the absolutely continuous parts of the spectra of H_0 and H_1 must be *identical*.

The next theorem concerns the « transitivity » of the wave operators. It plays an important rôle at various stages of our theory, though only a little use of it will be made in this paper. For instance, it is an indispensable tool in proving the existence theorem of wave operators (see (9)). The following theorem was first given by Kato (6), but the proof was only sketched briefly. Therefore, we restate it with the detailed proof because of its importance in the whole theory.

Theorem 3.2. *If* $W_+(H_1, H_0)$ *and* $W_+(H_2, H_1)$ *exist, then* $W_+(H_2, H_0)$ *also exists and*

$$W_+(H_2, H_0) = W_+(H_2, H_1)W_+(H_1, H_0)\,. \tag{3.10}$$

The same assertions hold true for W_- *in place of* W_+.

3·3. – We now give the proofs of these theorems in several steps.

Proof of i) *and* ii) *of Theorem* 3.1. *Since* U_t, $-\infty < t < +\infty$, as given by (3.1), are unitary, U_tP_0 are partially isometric operators with the initial set $\mathfrak{M}_0$. Since $W_+ = s - \lim_{t\to+\infty} U_tP_0$ by definition, we obtain, by making use of the continuity of the norm, $\|W_+u\| = \lim \|U_tP_0u\| = \|P_0u\|$, which means that W_+ is a partially isometric operator with the initial set $\mathfrak{M}_0$. Since P_0 commutes with $\exp[itH_0]$, we have $\exp[itH_1]U_sP_0 = U_{s+t}P_0 \exp[itH_0]$. By letting $s \to +\infty$ (t being fixed) on both sides of the last equality, we then obtain (3.5). As is well known, (3.5) implies (3.6); this is a simple consequence of the spectral theory of continuous unitary groups in a Hilbert space (see *e.g.* Riesz and Sz.-Nagy (11), pp. 380-383). From (3.6) we see that $W_+\mathfrak{H}$ reduces $E_1(\lambda)$ for all real λ. Therefore, $W_+\mathfrak{H}$ reduces H_1 (see Stone (10), Theorem 7.15). Since W_+ is partially isometric, (3.6) shows that the part of H_1 in $W_+\mathfrak{H}$ is unitarily equivalent to the part of H_0 in $\mathfrak{M}_0$ (see Stone (10), Theorem 7.1). Since the latter is absolutely continuous, it follows that the part of H_1 in $W_+\mathfrak{H}$ is also absolutely continuous and therefore that $W_+\mathfrak{H} \subset \mathfrak{M}_1$ by virtue of Lemma 2.1, ii). (3.7) follows immediately from the unitary equivalence established above (see (2.2)). Thus i) and ii) of Theorem 3.1 have been proved.

Before proceeding to the proof of iii) of Theorem 3.1, it is convenient to prove Theorem 3.2.

Proof of Theorem 3.2. As is well known, $s-\lim_{n\to\infty} A_n = A$ and $s-\lim_{n\to\infty} B_n = B$ imply $s-\lim_{n\to\infty} A_n B_n = AB$. By applying this to $W_+(H_2, H_1)$ and $W_+(H_1, H_0)$, we obtain from $(3.2)_+$

$$(3.11)\qquad W_+(H_2, H_1)W_+(H_1, H_0) = s - \lim_{t\to+\infty} \exp[itH_2]P_1 \exp[-itH_0]P_0 ,$$

where we used the fact that P_1 commutes with $\exp[itH_1]$. To prove (3.10), it suffices to show that the factor P_1 on the right side of (3.11) can be removed. Now, it follows from i) of Theorem 3.1 that $W_+(H_1, H_0)u \in \mathfrak{M}_1$ and $\|W_+(H_1, H_0)u\| = \|P_0 u\|$. Therefore, by applying the same results to the pair (H_2, H_1) and (H_1, H_0), we have $\|W_+(H_2, H_1)W_+(H_1, H_0)u\| = \|P_1 W_+(H_1, H_0)u\| = \|W_+(H_1, H_0)u\| = \|P_0 u\|$. Hence, it follows from (3.11) that

$$(3.12)\qquad \lim_{t\to\infty} \|P_1 \exp[-itH_0]P_0 u\| = \lim_{t\to\infty} \|\exp[itH_2]P_1 \exp[-itH_0]P_0 u\| = \|P_0 u\| .$$

On the other hand, we have

$$(3.13)\qquad \|P_1 \exp[-itH_0]P_0 u\|^2 + \|(I-P_1)\exp[-itH_0]P_0 u\|^2 = \|\exp[-itH_0]P_0 u\|^2 = \|P_0 u\|^2 .$$

From (3.12), (3.13) and the unitary property of $\exp[itH_2]$ we finally obtain

$$\lim_{t\to\infty} \|\exp[itH_2](I-P_1)\exp[-itH_0]P_0 u\|^2 = \lim_{t\to\infty} \|(I-P_1)\exp[-itH_0]P_0 u\|^2 = 0 ,$$

which shows that the factor P_1 on the right side of (3.11) can be removed. This completes the proof of Theorem 3.2.

Proof of iii) *of Theorem* 3.1. First we assume that $W_+(H_0, H_1)$ exists. Since it is obvious that $W_+(H_1, H_1) = P_1$, we have by (3.10) and (3.4)

$$\mathfrak{M}_1 = P_1\mathfrak{M}_1 = W_+(H_1, H_1)\mathfrak{M}_1 = W_+(H_1, H_0)W_+(H_0, H_1)\mathfrak{M}_1 \subset W_+(H_1, H_0)\mathfrak{H} \subset \mathfrak{M}_1 .$$

Thus all the signs of inclusion of the last relation can be replaced be the sign of equality. In particular, we have $W_+(H_1, H_0)\mathfrak{H} = \mathfrak{M}_1$, as we wished to show. Conversely, assume that $W_+(H_1, H_0)\mathfrak{H} = \mathfrak{M}_1$. Then for an arbitrary $u \in \mathfrak{M}_1$ there exists a $v \in \mathfrak{M}_0$ such that $u = s-\lim_{t\to+\infty} U_t(H_1, H_0)v$. From this it follows immediately that $v = s-\lim_{t\to+\infty} U_t(H_0, H_1)u$. Therefore $U_t(H_0, H_1)u$ has

a limit as $t \to +\infty$, provided $u \in \mathfrak{M}_1$, which means that $W_+(H_0, H_1) = s - \lim U_t(H_0, H_1)P_1$ exists. Finally, in such a case, (3.8) follows immediately from the relation $W_+(H_0, H_1)W_+(H_1, H_0) = W_+(H_0, H_0) = P_0$ and the fact that $W_+(H_0, H_1)$ is a partially isometric operator with the initial set $\mathfrak{M}_1$, which coincides with the final set of $W_+(H_1, H_0)$ (see Sect. **2**·2).

Proof of Corollary to Theorem 3.1. (3.9) follows from (3.7), namely $SH_0P_0 = W_+^*W_-H_0P_0 = W_+^*H_1P_1W_- = H_0P_0W_+^*W_- = H_0P_0S$. Other assertions of the corollary are direct consequences of the definition of S and i) of Theorem 3.1.

3·4. – It follows from iii) of Theorem 3.1 that the existence of the four wave operators $W_\pm(H_1, H_0)$ and $W_\pm(H_0, H_1)$ implies that $W_+(H_1, H_0)\mathfrak{H} = W_-(H_1, H_0)\mathfrak{H} = \mathfrak{M}_1$ and accordingly solves the problem (II). Hence, in order to solve the problems stated in Section **1**, it suffices to prove the existence of $W_\pm(H_1, H_0)$ and $W_\pm(H_0, H_1)$. This is the reason why we introduce the generalized wave operators. If we adopt the definition $W_\pm(H_0, H_1) = s - \lim \exp[itH_0] \exp[-itH_1]$, then the above situation can not be expected. This can be seen in the following manner. Even in the usual physical problems in which H_0 is absolutely continuous, H_1 need not be absolutely continuous, because H_1 may have point eigenvalues. In such a case, let u be an eigenvector of H_1; then $\exp[itH_0] \exp[-itH_1]u$ does not possess a limit in general and hence the wave operators $W_\pm(H_0, H_1)$ in the usual sense do not exist. We can avoid such a circumstance by introducing the generalized wave operators.

The direct proof of $W_+\mathfrak{H} = W_-\mathfrak{H}$ without any reference to the existence of $W_\pm(H_0, H_1)$ seems to be rather difficult. In this connection, however, the following theorem is useful.

THEOREM 3.3. *If $W_+(H_1, H_0)$ and $W_-(H_1, H_0)$ both exist and the absolutely continuous part of the spectrum of H_1 is simple, then $W_+\mathfrak{H} = W_-\mathfrak{H}$.*

Proof. We denote the part of H_1 in $\mathfrak{M}_1$ by H_1'. ($\mathfrak{M}_1$ is the absolutely continuous subspace of $\mathfrak{H}$ with respect to H_1). By the assumption H_1' has a simple spectrum. On the other hand, we see by virtue of Theorem 3.1 that $W_\pm\mathfrak{H}$ are contained in $\mathfrak{M}_1$ and reduce H_1. Hence $W_\pm\mathfrak{H}$ also reduce H_1' and the parts of H_1' in $W_\pm\mathfrak{H}$ are equal to the parts of H_1 in $W_\pm\mathfrak{H}$. Since the latter have the same spectrum, we see that the parts of H_1' in $W_\pm\mathfrak{H}$ have the same spectrum. Then the desired relation $W_+\mathfrak{H} = W_-\mathfrak{H}$ follows immediately from the general theorem stating that, if a self-adjoint operator H has a simple spectrum, then two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ each reducing H coincide with each other if and only if the parts of H in $\mathfrak{M}$ and $\mathfrak{N}$ have the same spectrum (see STONE [10], Theorem 7.16).

3·5. – Finally, for later use we shall state a lemma which gives a sufficient condition for the existence of $W_\pm$. The same result has also been given by

JAUCH and ZINNES (3). (See Theorem 2 and the argument just before the end of (3).) For convenience' sake, we shall restate it in the form of a lemma. The proof of it is essentially given in (3) and may be omitted.

LEMMA 3.1. *Let H_0 and H_1 be self-adjoint operators such that $\mathfrak{D}(H_0) \cap \mathfrak{D}(H_1)$ is dense in $\mathfrak{M}_0$ and let $V = H_1 - H_0$. Further, we assume that there exists a subset $\mathfrak{D}$ of $\mathfrak{M}_0$ which satisfies the following conditions*:

(3.14) *the linear manifold determined by $\mathfrak{D}$ is dense in $\mathfrak{M}_0$,*

$$\exp[-itH_0]u \in \mathfrak{D}(H_0) \cap \mathfrak{D}(H_1) = \mathfrak{D}(V) \tag{3.15}$$

for every $u \in \mathfrak{D}$ and every real t. Then, for the existence of $W_+(H_1, H_0)$, it is sufficient that the relation

$$\int_{t_0}^{\infty} \| V \exp[-itH_0]u \| \mathrm{d}t < \infty \tag{3.16}$$

holds for any $u \in \mathfrak{D}$ and for some real t_0 (t_0 may depend on u).

The same assertion holds for W_- and $-\infty$ in place of W_+ and $+\infty$ respectively.

4. – Scattering by a potential I.

4·1. – From now on, we shall be concerned with the scattering problem in elementary wave mechanics. We assume that the total Hamiltonian of the system is formally given by the differential operator $H_1 = -\Delta + V(x)\cdot$ in $L^2(E_m)$. Here $\Delta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$, and $V(x)$ is a real-valued, measurable function defined on E_m. Throughout the present section we assume that $|V(x)|^2$ is integrable in any *bounded* domain in E_m. We take the unperturbed Hamiltonian H_0 as $H_0 = -\Delta$ and examine under what conditions imposed on $V(x)$ the wave operators and the scattering operator exist.

For the mathematically rigorous consideration in which we are now engaged, however, the domain of the operators H_0 and H_1 must be clearly prescribed so that they become self-adjoint. This problem has been investigated by several authors (KATO (14) STUMMEL (15) and WIENHOLTZ (16)). Let H_0' be the differential operator $-\Delta$ in $L^2(E_m)$ restricted on the set of sufficiently smooth functions of $L^2(E_m)$ such that $\Delta u \in L^2(E_m)$. H_0' is a symmetric operator but not

(14) T. KATO: *Trans. Amer. Math. Soc.*, **70**, 195 (1951).
(15) F. STUMMEL: *Math. Ann.*, **132**, 150 (1956).
(16) E. WIENHOLTZ: *Math. Ann.*, **435**, 50 (1958).

self-adjoint. However, it was shown that H_0' has a unique self-adjoint extension ([14-16]). Based on this fact H_0 is now defined as follows.

(4.1) H_0 is the uniquely determined self-adjoint extension of H_0'.

This definition is known to be equivalent to the following definition in terms of Fourier transforms (see KATO ([14])), which will be frequently used in the sequel.

(4.2) $u \in \mathfrak{D}(H_0)$ if and only if $|k|^2\hat{u}(k)$ belongs to $L^2(E_m)$ and

$$(H_0u)\hat{}(k) = |k|^2\hat{u}(k) .$$

By means of this definition we see easily that H_0 has an absolutely continuous spectrum which extends from 0 to $+\infty$. Therefore $\mathfrak{M}_0 = \mathfrak{H}$ and $P_0 = I$.

The definition of the operator V is clear: its domain consists of all $u \in L^2(E_m)$ such that $|V(x)u(x)|^2$ is integrable and $(Vu)(x) = V(x)u(x)$.

The definition of H_1 is somewhat complicated, because the symmetric operator H_0+V may not be self-adjoint. However, since H_0+V is a real operator (*), it has at least one self-adjoint extension ([10], Theorem 9.14). (As for the uniqueness of the extension, see Remark 2 below Theorem 4.1.) In the sequel we take H_1 as an arbitrarily fixed self-adjoint extension of H_0+V.

After these preparations we now formulate our first theorem as follows.

THEOREM 4.1. *Let H_0 and V be self-adjoint operators defined as above and let H_1 be an arbitrary self-adjoint extension of H_0+V. Further, let there exist a positive number ε such that*

$$(4.3) \qquad V(x)(1+r)^{-(m/2-1)+\varepsilon} \in L^2(E_m) , \qquad r = |x| .$$

Then the wave operators $W_\pm(H_1, H_0)$ exist.

COROLLARY. *If in addition to the assumption of Theorem* 4.1 *we assume that* i) $m = 1$ *or* ii) $m \geqslant 2$ *and $V(x)$ is spherically symmetric* (i.e. $V(x)$ *depends only on $r = |x|$*), *then the relation $W_+\mathfrak{H} = W_-\mathfrak{H}$ holds true and hence, by virtue of Corollary* 3.1, *the scattering operator S is unitary.*

Remark 1. The condition (4.3) means, roughly speaking, that $|V(x)|^2$ is integrable in any bounded domain and falls of more rapidly than r^{-1} at infinity (irrespective of m). Moreover, when $m = 3$, the condition $V(x) \in L^2(E_3)$

(*) The meaning of the real operator will be clear. For the exact definition, see STONE ([10]), chapt. IX.

implies (4.3). Thus we see that Theorem 4.1 is really a generalization of the results of COOK ([2]) and JAUCH and ZINNES ([3]) mentioned in Section **1**.

Remark 2. From the physical point of view, H_0+V is expected to admit a unique self-adjoint extension. For otherwise the eigenvalue problem of the system will not be uniquely solved. In this connection some sufficient conditions which assure the uniqueness of the extension have been obtained by KATO ([14]), STUMMEL ([15]) and WIENHOLTZ ([16]). According to the results of these authors, (4.3) assures the uniqueness of the extension if $m \leqslant 3$. When $m > 3$, it seems to be open whether or not (4.3) assures the uniqueness of the extension.

Proof of Theorem 4.1. We apply Lemma 3.1 to the present case. For that purpose we must first determine the subset $\mathfrak{D}$ of $\mathfrak{M}_0 = \mathfrak{H}$ mentioned in that lemma. We define $\mathfrak{D}$ as the set of all functions φ_a such that their Fourier transforms are given by

$$\hat{\varphi}_a(k) = \Big(\prod_{i=1}^{m} k_i\Big) \exp[-k^2 - ika], \tag{4.4}$$

where a varies over all vectors of E_m and k_i is the i-th component of the vector k: $k = (k_1, ..., k_m)$ (*). We first prove that $\mathfrak{D}$ satisfies (3.14) and (3.15) with $\mathfrak{M}_0 = L^2(E_m)$. By virtue of (4.2) we have

$$(\exp[-itH_0]\varphi_a)\hat{}\,(k) = \exp[-itk^2]\hat{\varphi}_a(k)\,. \tag{4.5}$$

Accordingly, by making use of the theory of the Fourier transforms, we obtain from (4.4) that

$$\begin{aligned}(\exp[-itH_0]\varphi_a)(x) &= \frac{1}{(2\pi)^{m/2}}\int \hat{\varphi}_a(k) \exp[-itk^2 + ikx]\,dk = \\ &= \frac{1}{(2\pi)^{m/2}} \prod_{i=1}^{m} \int_{-\infty}^{\infty} k_i \exp[-(1+it)k_i^2 + ik_i(x_i - a_i)]\,dk_i = \\ &= \text{const} \prod_{i=1}^{m} \frac{x_i - a_i}{(1+it)^{\frac{3}{2}}} \exp\left[-\frac{(x_i-a_i)^2}{4(1+it)}\right] = \frac{\text{const}}{(1+it)^{3m/2}} \Big\{\prod_{i=1}^{m}(x_i - a_i)\Big\} \exp\left[-\frac{(x-a)^2}{4(1+it)}\right].\end{aligned} \tag{4.6}$$

(*) Instead of (4.4), JAUCH and ZINNES ([3]) choose the family of functions $\hat{\varphi}_{k_0}(k) = \exp[-\alpha^2(k-k_0)^2]$, where α is a real number. When $m \geqslant 3$, this choice of $\mathfrak{D}$ also does for the proof of the present theorem, but it is not adequate in the case $m=1$ or $m=2$.

In particular, if we put $t=0$ in (4.6) we get

$$\varphi_a(x) = \varphi(x-a)\,, \quad \varphi(x) = \text{const} \prod_{i=1}^{m} x_i \exp[-x^2/4]\,. \tag{4.7}$$

Since $\hat{\varphi}(k) = \prod_{i=1}^{m} k_i \exp[-k^2]$, $\hat{\varphi}(k)$ vanishes only in a set of measure zero. Therefore, (3.14) is a direct consequence of Wiener's theorem on the closure of translations (*) stating that, if the Fourier transform $\hat{\varphi}(k)$ of the function $\varphi(x) \in L^2(E_m)$ has a set of zeros of measure zero, then the family of functions $\{\varphi(x-a);\ a \in E_m\}$ determines a dense linear manifold in $L^2(E_m)$.

We next prove (3.15). It follows immediately from (4.4), (4.5) and (4.2) that $\exp[-itH_0]\varphi_a \in \mathfrak{D}(H_0)$. On the other hand (4.6) implies that $(1+r)^{m/2-1-\varepsilon} \cdot (\exp[-itH_0]\varphi_a)(x)$, $\varepsilon > 0$, is a bounded function. Therefore, we see from (4.3) that $V(x)(\exp[-itH_0]\varphi_a)(x)$ belongs to $L^2(E_m)$. This means that $\exp[-itH_0]\varphi_a \in \mathfrak{D}(V)$. Thus we obtain that $\exp[-itH_0]\varphi_a \in \mathfrak{D}(H_0) \cap \mathfrak{D}(V) \subset \subset \mathfrak{D}(H_0) \cap \mathfrak{D}(H_1)$, because H_1 is the extension of H_0+V by definition. This proves (3.15).

Now, by virtue of Lemma 3.1 we see that for the proof of Theorem 4.1 it suffices to prove the inequality

$$\int_{-\infty}^{+\infty} \| V \exp[-itH_0]\varphi_a \| \mathrm{d}t < \infty \tag{4.8}$$

for all φ_a. It follows from (4.6) that for any real δ

$$\begin{aligned} |(\exp[-itH_0]\varphi_a)(x)| &= \frac{\text{const}}{|1+it|^{3m/2}} \Big\{\prod_{i=1}^{m} |x_i - a_i|\Big\} \exp\left[-\frac{(x-a)^2}{4(1+t^2)}\right] \leqslant \\ &\leqslant \frac{\text{const}}{|1+it|^{3m/2}} |x-a|^m \exp\left[-\frac{(x-a)^2}{4(1+t^2)}\right] = \\ &= \text{const}\, \frac{|x-a|^{-(m/2-1)+\delta}}{|1+it|^{1+\delta}} \left[\left|\frac{x-a}{1+it}\right|^{3m/2-1-\delta} \exp\left[-\frac{(x-a)^2}{4(1+t^2)}\right]\right]. \end{aligned} \tag{4.9}$$

We now put $X = (x-a)/2(1+t^2)^{\frac{1}{2}}$ and take δ as $0 < \delta < \frac{1}{2}$, so that

(*) N. WIENER ([17]), Sect. **15**, Theorem 11.

([17]) N. WIENER: *The Fourier Integral* (Cambridge, 1933).

$3m/2-1-\delta>0$ for any positive integer m. Then it follows from (4.9) that

$$(4.10)\quad |(V\exp[-itH_0]\varphi_a)(x)| \leqslant$$

$$\leqslant \text{const}\,\frac{|V(x)(x-a)^{-(m/2-1)+\delta}|}{|1+it|^{1+\delta}}\,(2X)^{3m/2-1-\delta}\exp[-X^2]\leqslant$$

$$\leqslant M\,\frac{|V(x)(x-a)^{-(m/2-1)+\delta}|}{|1+it|^{1+\delta}}\,,\qquad 0<\delta<\tfrac{1}{2},$$

where M is a constant independent of x and t. Now we restrict δ further by $0<\delta<\min(\frac{1}{2},\varepsilon)$, where ε is the constant given in (4.3). Then it follows easily from (4.3) that, for any real t, the right side of (4.11) belongs to $L^2(E_m)$ as a function of x. Thus we finally obtain the estimate

$$\|V\exp[-itH_0]\varphi_a\|\leqslant N|1+it|^{-(1+\delta)}\,,$$

where N is a constant depending only on a. This shows that the inequality (4.8) holds true for all φ_a, as we wished to show. Theorem 4.1. is now completely proved.

Proof of Corollary to Theorem 4.1. When $m=1$, H_1 is nothing but an ordinary differential operator in $L^2(E_1)$. Under the condition (4.3) this operator is known to have a simple spectrum (KODAIRA ([18])). Therefore, the assertion follows directly from Theorem 3.3. To consider the remaining case in which $m\geqslant 2$ and $V(x)$ is spherically symmetric, we assume for brevity that $m=3$; the general case can be dealt with quite in the same way. Since $V(x)$ is spherically symmetric, we can separate the radial variable from the angular variables. More precisely, $\mathfrak{H}$ is decomposed as $\mathfrak{H}=\sum_{0\leq l<\infty,|m|\leq l}\oplus\mathfrak{H}_{lm}$, where $\mathfrak{H}_{lm}$ is a set of all functions of the form $r^{-1}u(r)Y_{lm}(\theta,\varphi)$, $u(r)\in L^2(0,\infty)$ ($Y_{lm}(\theta,\varphi)$ denote the spherical harmonics). Then each $\mathfrak{H}_{lm}$ reduces both H_0 and H_1 and therefore also $W_\pm$. As is well known, however, the parts $(H_0)_{lm}$ and $(H_1)_{lm}$ of H_0 and H_1 in $\mathfrak{H}_{lm}$ are unitarily equivalent to the ordinary differential operators in $L^2(0,\infty)$ given by

$$(4.11)\quad (H_0)_{lm}=-\mathrm{d}^2/\mathrm{d}r^2+l(l+1)r^{-2}\cdot\,,\quad (H_1)_{lm}=-\mathrm{d}^2/\mathrm{d}r^2+[l(l+1)r^{-2}+V(r)]\cdot\,.$$

When $l=0$, we must impose the boundary condition $u(0)=0$. (When $m>3$, the boundary condition at 0 may be necessary even if $l>0$.) It was shown by KODAIRA ([18]) that, under the condition (4.3), $(H_1)_{lm}$ has a simple spectrum. Again our assertion follows immediately from Theorem 3.3.

([18]) K. KODAIRA: *Amer. Journ. Math.*, **71**, 921 (1949).

4·2. – We next apply Theorem 4.1 to the scattering problem of the system consisting of s particles. We denote by $\boldsymbol{x}_i$, $i=1, ..., s$, the individual (3-dimensional) position vector of the i-th particle. The configuration space is a $3s$-dimensional Euclidean space E_{3s} and the vector x of the configuration space can be expressed as $x=(\boldsymbol{x}_1, ..., \boldsymbol{x}_s)$. We now assume that the potential $V(x)$ is given by

$$V(x)=V(\boldsymbol{x}_1, ..., \boldsymbol{x}_s) \tag{4.12}$$

$$=V_0(\boldsymbol{x}_1, ..., \boldsymbol{x}_s)+\sum_{i=1}^{s} V_{0i}(\boldsymbol{x}_i)+\sum_{1\leq i<j\leq s} V_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j)\,.$$

The meaning of each term will be clear. We take H_0 as in Section **4** (*) and examine the condition which assures the existence of $W_{\pm}(H_1, H_0)$, $H_1 \supset H_0+V$ (**). In this connection we have the following theorem.

THEOREM 4.2. *Let* $m=3s$, $s=1, 2, ...,$ *and let the function* $V(x)$ *defined in* E_{3s} *be given by* (4.12). *Further assume that* i) $V_0(\boldsymbol{x}_1, ..., \boldsymbol{x}_s)$ *satisfies* (4.3) *as a function of* $3s$ *variables* (i.e. *in* (4.3) *put* $m=3s$ *and* $r=|x|$), ii) $V_{0i}(\boldsymbol{x}_i)$, $i=1, ..., s$, and $V_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j)$, $1\leqslant i<j\leqslant s$, *satisfy* (4.3) *as a function of* 3 *variables* (i.e. *in* (4.3) *put* $m=3$ *and* $r=|\boldsymbol{x}_i|$ *or* $r=|\boldsymbol{x}_i-\boldsymbol{x}_j|$). *Then* $V(x)$ *satisfies* (4.3) *as a function of* $3s$ *variables and accordingly the assertion of Theorem* 4.1. *holds.*

Proof. It suffices to prove that each of the functions $V_0(x)$, $V_{0i}(\boldsymbol{x}_i)$ and $V_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j)$ satisfies (4.3) as a function of $3s$ variables. As for $V_0(x)$, this is assumed in the theorem. As for $V_{01}(\boldsymbol{x}_1)$, we proceed as follows. We set $r=|x|=(|\boldsymbol{x}_1|^2+...+|\boldsymbol{x}_s|^2)^{\frac{1}{2}}$, $r'=(|\boldsymbol{x}_2|^2+...+|\boldsymbol{x}_s|^2)^{\frac{1}{2}}$ and $r_1=|\boldsymbol{x}_1|$. Then for any δ such that $0<\delta<\frac{1}{3}$ we have

$$\int_{E_{3s}} |V_{01}(\boldsymbol{x}_1)(1+r)^{-(3s/2-1)+\delta}|^2\,\mathrm{d}x=\int_{E_{3s}} \frac{|V_{01}(\boldsymbol{x}_1)|^2}{(1+r)^{(-1)+\delta}(1+r)^{1-3\delta}}\,\mathrm{d}\boldsymbol{x}_1 ... \mathrm{d}\boldsymbol{x}_s \leqslant$$

$$\leqslant \int_{E_{3s}} \frac{|V_{01}(\boldsymbol{x}_1)|^2}{(1+r')^{3(s-1)+\delta}(1+r_1)^{1-3\delta}}\,\mathrm{d}\boldsymbol{x}_1 ... \mathrm{d}\boldsymbol{x}_s =$$

$$=\int_{E_3} |V_{01}(\boldsymbol{x}_1)(1+r_1)^{-\frac{1}{2}+3\delta/2}|^2 \mathrm{d}\boldsymbol{x}_1 \int_{E_{3(s-1)}} (1+r')^{-3(s-1)-\delta}\,\mathrm{d}\boldsymbol{x}_2 ... \mathrm{d}\boldsymbol{x}_s\,.$$

(*) In such a many-particle problem a different choice of H_0 may be possible, but we shall not pursue this problem here.

(**) In such a case the uniqueness of the extension H_1 was treated by KATO (14) with a satisfactory result.

The second integral on the right side is finite by $\delta > 0$. By the assumption of the theorem there exists a positive number ε such that $V_{01}(\boldsymbol{x}_1)(1+r_1)^{-\frac{1}{2}+\varepsilon} \in L^2(E_3)$. Therefore, if we take δ such that $0 < \delta < \min(2\varepsilon/3, \frac{1}{3})$, then the first integral on the right side is also finite. Thus we obtain that $V_{01}(\boldsymbol{x}_1)(1+r)^{-(3s/2-1)+\delta} \in L^2(E_{3s})$ for such δ. This means that $V_{01}(\boldsymbol{x}_1)$ satisfies (4.3) as a function of $3s$ variables. The functions $V_{0i}(\boldsymbol{x}_i)$, $i \neq 1$, can be treated in a similar manner. Next we consider the function $V_{12}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$. For that purpose, we introduce the orthogonal transformation of the co-ordinate system of E_{3s} by the following formula:

$$\boldsymbol{x}_1 - \boldsymbol{x}_2 = \sqrt{2}\boldsymbol{X}_1\,, \quad \boldsymbol{x}_1 + \boldsymbol{x}_2 = \sqrt{2}\boldsymbol{X}_2\,, \quad \boldsymbol{x}_i = \boldsymbol{X}_i\,, \quad 3 \leqslant i \leqslant s\,.$$

In the new co-ordinate system the function V_{12} depends only on $\boldsymbol{X}_1$ and by the assumption of the theorem $V_{12}(\boldsymbol{X}_1)(1+r)^{-\frac{1}{2}+\varepsilon} \in L^2(E_3)$ for some $\varepsilon > 0$. By the same argument as in the case of $V_{01}(\boldsymbol{x}_1)$ we can then prove that

$$V_{12}(\boldsymbol{X}_1)(1+R)^{-(3s/2-1)+\delta} \in L^2(E_{3s})\,, \quad 0 < \delta < \min(2\varepsilon/3, \tfrac{1}{3}),$$

where $R = |X| = (|\boldsymbol{X}_1| + \ldots + |\boldsymbol{X}_s|)^{\frac{1}{2}}$. Since the transformation of the co-ordinate is orthogonal, this implies that $V_{12}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ satisfies (4.3) as a function of $3s$ variables. Other V_{ik} are treated in the same way. Thus Theorem 4.2 is completely proved.

5. – Scattering by a potential II.

5·1. – In the previous Section we proved the existence of the wave operators under the assumption (4.3) imposed on the potential $V(x)$. Under that condition alone, however, we have not been able to solve the problem (II). In other words, we do not know in general whether $W_+\mathfrak{H}$ coincides with $W_-\mathfrak{H}$ or not. In this Section we prove the relation $W_+\mathfrak{H} = W_-\mathfrak{H}$, together with the existence of $W_\pm$, under somewhat stronger restrictions on V. As mentioned in Section **1**, we first state two lemmas to be used in the sequel.

To this end we need the notion of the *Schmidt class* of operators in a general Hilbert space (see *e.g.* STONE [10], Chapt. II and III). Let $\{\varphi_\nu\}$ be a complete orthonormal set of $\mathfrak{H}$. Then for any bounded operator A the number

$$\|A\|_2 = (\sum_\nu \|A\varphi_\nu\|^2)^{\frac{1}{2}}$$

is independent of the choice of $\{\varphi_\nu\}$. $\|A\|_2$ is usually called the *Schmidt norm* of A. (It may be infinite). The set of all bounded operators with finite Schmidt norm is called the Schmidt class of operators. We denote it by $\boldsymbol{S}$. When $\mathfrak{H}$

is the function space $L^2(E_m)$, a bounded operator A belongs to the Schmidt class if and only if it can be represented as an integral operator with the kernel of Hilbert-Schmidt type, *i.e.* if there exists a function $k(x, y)$ in $E_m \times E_m$ such that $\iint |k(x, y)|^2 dx dy$ is finite and $(Au)(x) = \int k(x, y) u(y) dy$ for any $u \in L^2(E_m)$. Furthermore, we have

$$\|A\|_2^2 = \iint |k(x, y)|^2 dx dy . \tag{5.1}$$

In terms of the Schmidt class the above mentioned lemmas can be stated in the following form (for the proof, see KURODA (9)).

LEMMA 5.1. *Let H_0 and V be self-adjoint operators in a Hilbert space $\mathfrak{H}$. Let*

$$\mathfrak{D}(V) \supset \mathfrak{D}(H_0) \quad \text{and} \quad \|Vu\| \leq a\|H_0 u\| + b\|u\|, \quad \text{for any } u \in \mathfrak{D}(H_0), \tag{5.2}$$

where a and b are constants such that $0 \leq a < 1$ and $0 \leq b$. Furthermore, let

$$|V|^{\frac{1}{2}}(H_0 + i)^{-1} \in \boldsymbol{S} . \tag{5.3}$$

Then $H_1 = H_0 + V$ is also self-adjoint and the generalized wave operators $W_\pm(H_1, H_0)$ and $W_\pm(H_0, H_1)$ exist.

LEMMA 5.2. *Let H_0 be a fixed self-adjoint operator in $\mathfrak{H}$ and let V and V_n, $n = 1, 2, \ldots$, be self-adjoint operators satisfying the conditions (5.2) and (5.3) with constants a, b independent of n. Further assume that $V - V_n$, $n = 1, 2 \ldots$, have self-adjoint extensions $(V - V_n)^\sim$ satisfying the following conditions:*

$$|(V - V_n)^\sim|^{\frac{1}{2}}(H_0 + i)^{-1} \in \boldsymbol{S} , \tag{5.4}$$

$$\lim_{n \to \infty} \| |(V - V_n)^\sim|^{\frac{1}{2}}(H_0 + i)^{-1} \|_2 = 0 . \tag{5.5}$$

If we put

$$H_1 = H_0 + V , \qquad H^{(n)} = H_0 + V_n , \qquad S(H) = W_+(H, H_0)^* W_-(H, H_0) , \tag{5.6}$$

then

$$s - \lim_{n \to \infty} W_\pm(H^{(n)}, H_0) = W_\pm(H_1, H_0) , \quad s - \lim_{n \to \infty} S(H^{(n)}) = S(H_1) . \tag{5.7}$$

5·2. – The application of Lemma 5.1 to the differential operator $-\Delta + V$ leads to the following theorem.

THEOREM 5.1 (*). *Let $\mathfrak{H} = L^2(E_m)$, $m \leqq 3$, and let H_0 and V be self-adjoint operators in $L^2(E_m)$ defined in* Section **4**. *Furthermore, we assume that the potential $V(x)$ satisfies*

$$V(x) \in L^1(E_m) \cap L^2(E_m) . \tag{5.8}$$

Then $H_1 = H_0 + V$ is a self-adjoint operator and the generalized wave operators $W_\pm(H_1, H_0)$ and $W_\pm(H_0, H_1)$ exist. Accordingly, by virtue of Theorem 3.1 *and Corollary to Theorem* 3.1 *the scattering operator S is unitary.*

Proof. By virtue of Lemma 5.1 it suffices to show that H_0 and V satisfy (5.2) and (5.3). It was shown by KATO ([14]) that, if the function $V(x)$ belongs to $L^2(E_m)$, then H_0 and V satisfy (5.2). Therefore we have only to prove the relation (5.3). It follows from (4.2) that $\{(H_0 + i)^{-1}u\}\hat{}\,(k) = (k^2 + i)^{-1}\hat{u}(k)$. Hence we get

$$((H_0 + i)^{-1}u)(x) = (2\pi)^{-m/2}\int \frac{\exp[ikx]}{k^2 + i}\,\hat{u}(k)\,\mathrm{d}k . \tag{5.9}$$

Since $m \leqq 3$ by the assumption, $\exp[ikx](k^2 + i)^{-1} \in L^2(E_m)$. Therefore we can apply the Parseval's formula to the integral on the right side of (5.9), with the result that

$$((H_0 + i)^{-1}u)(x) = \int k(x - y)\,u(y)\,\mathrm{d}y , \tag{5.10}$$

where $k(z)$ is defined by

$$k(z) = (2\pi)^{-m}\int \frac{\exp[ikz]}{k^2 + i}\,\mathrm{d}k . \tag{5.11}$$

(5.11) shows that $k(z)$ is the Fourier transform of the function $(2\pi)^{-m/2}(k^2 + i)^{-1}$ belonging to $L^2(E_m)$. Hence $k(z)$ itself belongs to $L^2(E_m)$. On the other hand, it is obvious that $|V|^{\frac{1}{2}}$ is the multiplicative operator: $(|V|^{\frac{1}{2}}u)(x) = |V(x)|^{\frac{1}{2}}u(x)$. Hence we get from (5.10) that

$$(|V|^{\frac{1}{2}}(H_0 + i)^{-1}u)(x) = \int |V(x)|^{\frac{1}{2}}\,k(x - y)\,u(y)\,\mathrm{d}y . \tag{5.12}$$

(*) Quite recently, T. IKEBE ([19]) proved a theorem somewhat stronger than Theorem 5.1 by a different method. His proof is based on the eigenfunction expansions associated with the differential operator $-\Delta + V$.

([19]) T. IKEBE: *Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory*, to be published.

Thus, the operator $|V|^{\frac{1}{2}}(H_0+i)^{-1}$ is an integral operator with the kernel $|V(x)|^{\frac{1}{2}}k(x-y)$. Since $V(x)\in L^1(E_m)$ by the assumption of the theorem and $k(z)\in L^2(E_m)$ as is shown above, we have

$$\iint |V(x)|\,|k(x-y)|^2\,dx\,dy = \int |V(x)|\,dx \int |k(z)|^2 dz < \infty . \tag{5.13}$$

This means that the kernel $|V(x)|^{\frac{1}{2}}k(x-y)$ is of Hilbert-Schmidt type. Thus we obtain (5.3), as we wished to show.

5·3. – Next we consider the continuity property of the wave operator and the scattering operator with respect to V. By applying Lemma 5.2 to the differential operator $-\Delta+V$, we have the following theorem.

THEOREM 5.2. *Let the function $V(x)$ and $V_n(x)$, $n=1, 2, \ldots$, satisfy the condition* (5.8) *and let*

$$\lim_{n\to\infty} \int |V_n(x) - V(x)|\,dx = 0 . \tag{5.14}$$

We further assume that V and V_n satisfy (5.2) *with the constants a, b independent of n* (*). *Define H_1, $H^{(n)}$ and $S(H)$ as in* (5.6). *Then*

$$s-\lim_{n\to\infty} W_{\pm}(H^{(n)}, H_0) = W_{\pm}(H_1, H_0) ,$$

$$s-\lim_{n\to\infty} S(H^{(n)}) = S(H_1) .$$

Proof. By virtue of Lemma 5.2 it suffices to show that H_0, V and V_n satisfy all the assumptions of Lemma 5.2. It follows from Theorem 5.1 and the assumptions of the present theorem that V and V_n satisfy (5.2) and (5.3) with constants a and b independent of n. As is easily seen, $V-V_n$ has a unique self-adjoint extension $(V-V_n)^{\sim}$ which is the multiplicative operator: $((V-V_n)^{\sim}u)(x) = (V(x)-V_n(x))\,u(x)$. (Note that this is not always equal to $V-V_n$.) Since $V(x)-V_n(x)$ belongs to $L^1(E_m)$, it follows by the same argument as in the proof of Theorem 5.1 that (5.4) holds true and that

$$\| |(V-V_n)^{\sim}|^{\frac{1}{2}}(H_0+i)^{-1}|_2^2 = \int |V(x)-V_n(x)|\,dx \int |k(z)|^2\,dz .$$

(*) For this it is sufficient, for instance, that $\int |V_n(x)|^2 dx$ is bounded in n. We can see this by examining the method of proof of Lemma 4 of KATO (14).

(see (5.13)). By the assumption of the theorem the first integral on the right side tends to 0 as $n \to \infty$. This implies (5.5). Thus all the assumptions of Lemma 5.2 are satisfied and Theorem 5.2 is proved.

6. – Remarks on the nature of the spectrum of the operator $-\Delta + V$.

As is shown in Section **3**, the theory of wave operators is closely connected with the perturbation theory of continuous spectra of self-adjoint operators. In particular, from the results of the previous sections we can directly obtain some information on the nature of the spectrum of a partial differential operator $H_1 = -\Delta + V(x)\cdot$. We defined in Section **4** the operator H_1 as a self-adjoint extension of the symmetric operator $H_0 + V$. In the language of differential operators, it can also be said that H_1 is a partial differential operator $-\Delta + V$ with some *boundary conditions*. (In case $H_0 + V$ has a unique self-adjoint extension, we may say that no boundary condition is required.)

Theorems 3.1 and 4.1 now show that, if the function $V(x)$ satisfies the assumption of Theorem 4.1, the continuous spectrum of the operator H_1 contains a part with the same structure as the spectrum of H_0. ($H_0 \cong -\Delta$ as defined in Section **4**.) In particular, each non-negative number λ belongs to the continuous spectrum of H_1 and the multiplicity of λ is infinite, whatever the boundary condition may be. This is a partial answer to the question raised by E. C. TITCHMARSH in his book (TITCHMARSH [20], p. 158).

In case the assumption of Theorem 5.1 is satisfied, we have the stronger result that the absolutely continuous part of the spectrum of H_1 is unitarily equivalent to the spectrum of H_0. In this case we can also say that the part of H_1 in $\mathfrak{H} \ominus W_\pm \mathfrak{H}$ has a singular spectrum, *i.e.* a point spectrum plus, possibly, a singular continuous spectrum.

On the other hand, it is important for physical application to see whether or not the part of H_1 in $\mathfrak{H} \ominus W_\pm \mathfrak{H}$ has a pure point spectrum, that is, whether or not the singular continuous spectrum is absent. On this point the present method does not give any information. However, POVZNER [21] and IKEBE [19] proved the absence of the singular continuous spectrum under a condition almost equivalent to the one stated in Theorem 5.1. In such a case, therefore, Theorem 5.1, combined with their results, gives a positive answer to the above question.

([20]) E. C. TITCHMARSH: *Eigenfunction expansions associated with second-order differential equations*, part II (Oxford, 1958).

([21]) A. JA. POVZNER: *Mat. Zbornik*, **32** (**74**), 110 (1953), in Russian.

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Note added.

1. – Prof. M. N. HACK has kindly informed the writer of his paper: *Nuovo Cimento*, **9**, 731 (1958). In this paper he proved the same result as in Theorem 4.1 in the case of $m = 3$.

2. – In connection with Theorem 4.2 we can construct a simple example in which $W_{\pm}(H_1, H_0)$ exist and yet $W_{\pm}\mathfrak{H}$ do not coincide with $\mathfrak{M}_1$. To this end let $s = 2$ and let $V(\boldsymbol{x}_1, \boldsymbol{x}_2) = V_1(\boldsymbol{x}_1) + V_1(\boldsymbol{x}_2)$, where $V_1(\boldsymbol{x})$ satisfies (4.3) as a function of 3 variables. Furthermore, assume that the operator $-\Delta + V_1(x)$ in $L^2(E_3)$ has at least one negative eigenvalue. (For example, we may take $V(x)$ as a square well potential with a bound state.) Let λ be its smallest eigenvalue. Then the absolutely continuous spectrum of $-\Delta + V(\boldsymbol{x}_1, \boldsymbol{x}_2)$ extends from λ to $+\infty$. From this it follows immediately that $W_{\pm}\mathfrak{H}$ are proper subsets of $\mathfrak{M}_1$.

RIASSUNTO (*)

Si esamina il carattere matematico della teoria dello scattering secondo la formulazione che tien conto della « dipendenza dal tempo » con speciale accento sugli operatori d'onda e di scattering. Gli argomenti principalmente trattati sono i seguenti: 1) si riassumono alcune proprietà generali di detti operatori; 2) si prova l'esistenza dell'operatore d'onda con restrizioni sul potenziale relativamente poco severe; 3) si provano l'esistenza e l'unitarietà dell'operatore di scattering quando siano imposte alcune condizioni addizionali. L'argomentazione è matematicamente rigorosa.

(*) *Traduzione a cura della Redazione.*

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Dispersion Relations for Associated Production.

Y. S. Jin

Swiss Federal Institute of Technology - Zürich

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Summary. — Formal derivation of relativistic dispersion relations for the associated production of strange particles by pion-nucleon collision is presented, however, without trying to give rigorous proof of derivation concerning the analytic continuation of the causal amplitudes into the unphysical region. Matrix structure of causal amplitudes in spin and isospin space is studied. Dispersion relations are obtained in covariant form as well as in Breit system separately for spin-flip or non-spin-flip and isospin-flip or non-isospin-flip processes of the associated production.

1. – Introduction.

During the last few years, dispersion relations have been derived for various processes [1], either in a formal way or even with rigorous proof, concerning the analytic continuation of the casual amplitudes into the unphysical region, and have been successfully applied to pion-nucleon scattering etc. Recently, Amati and Vitale [2] derived dispersion relations for the processes involving baryons, pions and K-mesons, however, in the static approximation with fixed extended source.

In this paper, we try to derive relativistic dispersion relations for the associated production of strange particles by pion-nucleon collision only in a formal

(1) For instance, see, M. L. Goldberger: *Proc. of 1958 Annual Intern. Conf. on High Energy Physics at CERN* (Geneva, 1958), p. 207.

(2) D. Amati and B. Vitale: *Nuovo Cimento*, **6**, 1282 (1957).

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way, without touching the problem of the justification of the analytic continuation of causal amplitudes into the unphysical region, in the hope that it could be proved in the near future and it would turn out to be of much physical significance for practical application as in the cases which have been so far investigated. In fact, we restrict ourselves rather to getting kinematical orientations and intuitive aspects over the problem, especially, because there appear various strong couplings simultaneously, in addition to the kinematical complication due to the inelastic nature of the relevant process.

2. – Kinematical preliminaries.

Let the four-momenta of incoming and outgoing bosons be k and l, respectively, while those of fermions be p and q (3). Then the over-all energy-momentum conservation states:

$$k+p=l+q\,. \tag{2.1}$$

In virtue of the mass shell conditions, $k^2=-\mu_1^2$, $l^2=-\mu_2^2$, $p^2=-M_1^2$ and $q^2=-M_2^2$, there can be only two independent scalars, which we may choose as follows:

$$\nu=-(p+q)(k+1)\,, \qquad Q^2=(p-q)^2\,. \tag{2.2}$$

As usual, we introduce the Breit system, where

$$\boldsymbol{q}=-\boldsymbol{p}=\frac{\boldsymbol{\Delta}}{2}\,. \tag{2.3}$$

Unlike in the case of elastic scattering where the Breit system enjoys three major advantages over other frames, *i.e.*, $\boldsymbol{p}+\boldsymbol{q}=0$, $k_0=l_0$ and $(\boldsymbol{k}+\boldsymbol{l})(\boldsymbol{q}-\boldsymbol{p})=0$, in our cases it turns out to be $k_0\neq l_0$ and $(\boldsymbol{k}+\boldsymbol{l})(\boldsymbol{q}-\boldsymbol{p})\neq 0$. Fixing the Lorentz frame, the four-momenta in the Breit system are as follows:

$$\begin{cases} p=\left(\sqrt{M_1^2+\dfrac{\Delta^2}{4}};\,0,\,0,\,-\dfrac{\Delta}{2}\right), & q=\left(\sqrt{M_2^2+\dfrac{\Delta^2}{4}};\,0,\,0,\,\dfrac{\Delta}{2}\right), \\ k=\left(\omega+\dfrac{\varepsilon}{2};\,\pi_x,\,0,\,\pi_z+\dfrac{\Delta}{2}\right), & l=\left(\omega-\dfrac{\varepsilon}{2};\,\pi_x,\,0,\,\pi_z-\dfrac{\Delta}{2}\right), \end{cases} \tag{2.4}$$

(3) In this, note, we use the notations $p=(p_0=-ip_4,\boldsymbol{p})$ and $(p\cdot q)=(\boldsymbol{p}\cdot\boldsymbol{q})-p_0q_0$ for any four-vector.

where

$$k_0 - l_0 \equiv \varepsilon\,, \qquad p_0 + q_0 \equiv 2E\,, \qquad \boldsymbol{k} + \boldsymbol{l} \equiv 2\boldsymbol{\pi}\,.$$

From (2.4), it follows

$$\text{(2.5)} \qquad \begin{cases} \pi_z = \dfrac{1}{\Delta}\left(\varepsilon\omega + \dfrac{\mu_2^2 - \mu_1^2}{2}\right), \\ \pi^2 = |\boldsymbol{\pi}|^2 = \omega^2 - \omega_p^2\,, \end{cases}$$

with

$$\omega_p^2 \equiv \frac{1}{4}(\Delta^2 - \varepsilon^2) + \frac{\mu_1^2 + \mu_2^2}{2}\,.$$

In terms of the quantities (ω, Δ) in the Breit system, two independent scalars turn out to be

$$\text{(2.6)} \qquad \nu = 4E\omega\,, \qquad Q^2 = \Delta^2 - \varepsilon^2\,.$$

As we shall see later, it will be immediate to translate the dispersion relations in the Breit system into covariant form, since ν is homogeneous linear in ω.

As a preliminary for the later study of matrix structure of the causal amplitudes in spin space, we now derive some useful identities for the matrix elements. Assuming that the initial and final fermions are in their positive energy state, these states $u(p)$ and $v(q)$ are the positive energy solutions of the following Dirac equations,

$$\text{(2.7)} \qquad \begin{cases} \{ip + M_1\}u^\lambda(p) = 0\,, & (\lambda = 1, 2) \\ \{iq + M_2\}v^\lambda(q) = 0\,, & (p \equiv \gamma \cdot p) \end{cases}$$

with the normalizations

$$\text{(2.8)} \qquad \bar{u}^{\lambda'} u^\lambda = \delta_{\lambda'\lambda}\,, \qquad \bar{v}^{\lambda'} v^\lambda = \delta_{\lambda'\lambda}\,.$$

The explicit forms of $u(p)$ and $v(q)$ can be written as

$$\text{(2.9)} \qquad u_\alpha^\lambda(p) = \frac{\{-ip + M_1\}_{\alpha\lambda}}{\{2M_1(M_1 + p_0)\}^{\frac{1}{2}}}\,, \qquad \bar{v}_\alpha^\lambda(q) = \frac{\{-iq + M_2\}_{\lambda\alpha}}{\{2M_2(M_2 + q_0)\}^{\frac{1}{2}}}\,.$$

Now by making use of the explicit representation of γ [4], we obtain the fol-

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[4] We use the representation,

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_5 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \bar{u} = u^\dagger\gamma_4$$

lowing identities:

$$\bar{v}^{\lambda'}(q)u^{\lambda}(p) = N(M_2 + q_0)(M_1 + p_0)\delta_{\lambda'\lambda} - N(\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\boldsymbol{\sigma}\cdot\boldsymbol{q})_{\lambda'\lambda}\,, \tag{2.10a}$$

$$\begin{aligned} i\bar{v}^{\lambda}(q)\frac{k+l}{2}u(p) = \frac{N}{2}\big[&\{(M_2+q_0)\boldsymbol{p} + (M_1+p_0)\boldsymbol{q}\}(\boldsymbol{k}+\boldsymbol{l}) - \\ &- (k_0+l_0)\{(M_2+q_0)(M_1+p_0) + \boldsymbol{p}\cdot\boldsymbol{q}\}\big]\delta_{\lambda'\lambda} + \\ &+ i\frac{N}{2}\boldsymbol{\sigma}_{\lambda'\lambda}\cdot\big[(\boldsymbol{k}+\boldsymbol{l})\times\{\boldsymbol{p}(M_2+q_0) - \boldsymbol{q}(M_1+p_0)\} + \\ &+ (k_0+l_0)\boldsymbol{p}\times\boldsymbol{q}\big]\,, \end{aligned} \tag{2.10b}$$

where the normalization factor is defined by

$$N \equiv \big\{4M_1M_2(M_1+p_0)(M_2+q_0)\big\}^{-\frac{1}{2}}\,. \tag{2.11}$$

The invariant scalars ν, Q^2 can be expressed in terms of the total energy W and scattering angle θ as follows

$$\left\{\begin{array}{l} \nu = 2W^2 - (M_1^2 + M_2^2 + \mu_1^2 + \mu_2^2) - Q^2\,, \\ Q^2 = -\mu_1^2 - \mu_2^2 - 2|\boldsymbol{k}^c||\boldsymbol{l}^c|\cos\theta + 2k_0^c l_0^c\,, \end{array}\right. \tag{2.12}$$

where the superscript c designates the quantities in c.m.s.

3. – Causal amplitude and crossing process.

We consider the associated production of Σ-hyperon and K-meson by pion-nucleon collision. As to the Λ-hyperon production, we shall present only the resulting dispersion relations, since there is essentially no difference between two cases except for the matrix structure in isospin space. Let the spin and isospin labels of nucleon, Σ-hyperon, pion and K-meson be λ_1, λ_2, α and ν, respectively. In our consideration we neglect electromagnetic interaction and weak interaction, so that the mass difference among iso-multiplets is also discarded. According to LEHMANN *et al.* [5], the relevant S-matrix elements turn out to be

$$\begin{aligned} \langle q, \lambda_2; l, \nu|S|p, \lambda_1; k, \alpha\rangle = \frac{i}{(2\pi)^3}\left(\frac{\mu_1\mu_2}{k_0 l_0}\right)^{\frac{1}{2}}\int \mathrm{d}^4x \int \mathrm{d}^4y \cdot \\ \cdot \exp\left[-i(lx - ky)\right]K_x K_y \langle q, \lambda_2|\theta(x_0 - y_0)[\Phi_\nu^\dagger(x), \varphi_\alpha(y)]|p, \lambda_1\rangle\,, \end{aligned} \tag{3.1}$$

[5] H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **6**, 319 (1957).

where $\Phi_\nu(y)$ and $\varphi_\alpha(x)$ are pion and K-meson field-operators in Heisenberg picture, respectively, while K is the Klein-Gordon operator. The interaction currents of pion and K-meson are defined respectively by

$$\text{(3.2)} \qquad \begin{cases} K_x\varphi_\alpha(x) \equiv (\mu_1^2 - \Box_x^2)\varphi_\alpha(x) \equiv j_\alpha(x) , \\ K_y\Phi_\nu(y) \equiv (\mu_2^2 - \Box_y^2)\Phi_\nu(y) \equiv J_\nu(y) . \end{cases}$$

Here $|\rangle$ denotes the « incoming state ». Making use of the translational invariance and carrying out the integration in $\frac{1}{2}(x+y)$, we obtain an important factor $\delta(p+k-q-l)$, *i.e.*, energy-momentum conservation. Its coefficient with appropriate normalization factor, *i.e.*,

$$\text{(3.3)} \qquad M_{\nu\alpha}(l+k;\, q,\, p;\, \lambda_2,\, \lambda_1) \equiv i(2\pi)^3 \left(\frac{p_0 q_0}{M_1 M_2}\right)^{\frac{1}{2}} \int \mathrm{d}^4x \exp\left[-i(k+l)\cdot\frac{x}{2}\right] \cdot$$

$$\cdot \langle q,\, \lambda_2 | \left\{ \theta(x_0) \left[J_\nu^\dagger\left(\frac{x}{2}\right),\, j_\alpha\left(-\frac{x}{2}\right)\right] - i\delta(x_0)\left[J_\nu^\dagger\left(\frac{x}{2}\right),\, \dot{\varphi}_\alpha\left(-\frac{x}{2}\right)\right]\right\} | p,\, \lambda_1 \rangle ,$$

is usually called *causal amplitude*. The second term turns out to be an arbitrary function of p and q independent of $k+l$. The local nature of the theory and the appearence of $\theta(x_0)$ give rise to *a polynomial in $k+l$ with arbitrary coefficients* which are functions of p and q, *i.e.*, $P(k+l;\, p,\, q)$, and the contribution from the second term can be included in it. In fact, the polynomial P is very important in physical arguments concerning high energy behavior of the causal amplitude, however, the derivation of dispersion relations is essentially not affected by its existence except for minor modifications which necessitate a « *subtraction procedure* », though the resulting dispersion relations contain less information than those without subtraction. With this understanding, henceforth we suppress the polynomial P including the second term in (3.3).

As it is more illustrative to work in a conveniently defined definite frame than to stick throughout to the covariant form, we shall work in the Breit system. The final results can be straightforwardly translated into covariant form. In the Breit system, the causal amplitude turns out to be

$$\text{(3.4)} \qquad M_{\nu\alpha}(\omega, \boldsymbol{\Delta};\, \lambda_2,\, \lambda_1) = i(2\pi)^3 \left(\frac{p_0 q_0}{M_1 M_2}\right)^{\frac{1}{2}} \int \mathrm{d}x_0 \exp\left[i\omega x_0\right] \int \mathrm{d}^3\boldsymbol{x} \cdot$$

$$\cdot \exp\left[-i\boldsymbol{\pi}\cdot\boldsymbol{x}\right] \langle \boldsymbol{\Delta}/2,\, \lambda_2 \left| \theta(x_0)\left[J_\nu^\dagger\left(\frac{x}{2}\right),\, j_\alpha\left(-\frac{x}{2}\right)\right]\right| -\boldsymbol{\Delta}/2,\, \lambda_1 \rangle .$$

If the momentun-transfer $\boldsymbol{\Delta}$ (equivalently Q^2) is kept fixed, M can be regarded as a function of $\omega(\nu)$, through the ω-dependence of $\boldsymbol{\pi}$. In order to derive dis-

persion relations for fixed momentum-transfer Δ, we have to be able to continue M as a function of ω analytically into the whole upper half-plane of ω.

From (2.5) we see that π_x becomes imaginary for a certain region of real ω, while π_z is always real for any real value of ω. This very fact gives rise to the unphysical region where the integral (3.4) is no longer a well-defined quantity and it causes major difficulty in proving the dispersion relations. We now define $\pi_x(\omega)$ in the following manner:

$$\pi_x(\omega) = \begin{cases} \sqrt{a(\omega-\omega_+)(\omega+\omega_-)}\,, & \text{for } \omega_+ < \omega \\ i\sqrt{a(\omega_+-\omega)(\omega+\omega_-)}\,, & \text{for } -\omega_- < \omega < \omega_+ \text{ (unphysical region)} \\ -\sqrt{a(\omega-\omega_+)(\omega+\omega_-)}\,, & \text{for } -\omega_- > \omega, \end{cases} \tag{3.5}$$

where $\omega_\pm$ and a are given by

$$\begin{cases} \omega_\pm = (2Q^2)^{-1}\cdot\{\pm\varepsilon(\mu_2^2-\mu_1^2) + \Delta\sqrt{\{Q^2+(\mu_1^2+\mu_2^2)\}^2 - 4\mu_1^2\mu_2^2}\}, \\ a \;\; = Q^2/\Delta^2\,. \end{cases} \tag{3.7}$$

Because the change of the sign of a leads to interchange of physical and unphysical regions, we assume that a is positive, *i.e.*, $(p-q)$ is *space-like*. In terms of Δ, this restriction turns out to be

$$\Delta^2 > \frac{(M_2^2-M_1^2)^2}{2(M_1^2+M_2^2)}\,. \qquad (\text{or } Q^2>0). \tag{3.8}$$

Now we introduce the *crossing amplitude* in order to give physical significance to the causal amplitude for negative ω. The crossing process can be defined as follows

$$\begin{cases} \text{original process} & \pi + \mathrm{N} \to \Sigma + \mathrm{K} \\ \text{crossing process} & \pi^+ + \Sigma \to \mathrm{N} + \overline{\mathrm{K}} \end{cases}\,. \tag{3.8}$$

The latter process is not yet experimentally accessible, however, by invoking PT-invariance (space-time refection) of the strong interaction the causal amplitude for this process turns out to be same as that of $\mathrm{N}+\overline{\mathrm{K}} \to \Sigma+\pi$ The corresponding kinematical quantities for the crossing process can be directly obtained from those for the original process, just through the interchange of p and q (consequently M_1 and M_2), *i.e.*, the crossing of two fermion lines. Thus $\boldsymbol{\pi}'(\omega)$ (the prime is to show the different ω-dependence in the crossing process) appears to be

$$\boldsymbol{\pi}'(\omega) = -\boldsymbol{\pi}(-\omega)\,. \tag{3.9}$$

Now the causal amplitude for the crossing process can be written down

$$(3.10)\qquad N_{\nu\alpha}(\omega;-\boldsymbol{\Delta};\lambda_1,\lambda_2)=i(2\pi)^3\left(\frac{p_0q_0}{M_1M_2}\right)^{\frac{1}{2}}\int \mathrm{d}x_0\exp[i\omega x_0]\int \mathrm{d}^3\boldsymbol{x}\cdot$$

$$\cdot\exp[-i\boldsymbol{\pi}'\cdot\boldsymbol{x}]\,\langle -\boldsymbol{\Delta}/2,\lambda_1\,|\theta(x_0)\left[J_\nu\left(\frac{x}{2}\right),j^\dagger_\alpha\left(-\frac{x}{2}\right)\right]|\,\boldsymbol{\Delta}/2,\lambda_2\rangle\,.$$

From (3.9) and (3.10), we obtain the *crossing symmetry*:

$$(3.11)\qquad M_{\nu\alpha}(-\omega;\boldsymbol{\Delta};\lambda_2,\lambda_1)=N^*_{\nu\alpha}(\omega;-\boldsymbol{\Delta};\lambda_1,\lambda_2)\,.$$

4. – Matrix structure of the causal amplitude.

Making the usual assumptions concerning spin and isospin of K-meson and Σ-hyperon, there appears rather complete analogy between the amplitude for pion-nucleon scattering and that for our process considered with respect to its matrix structure in spin and isospin space. We also assume the *charge independence* of the strong interaction (6), *i.e.*, invariance under the three-dimensional rotation in isospin space. Denoting the isospin variables of pion and Σ-hyperon by α and β, respectively, we get following matrix expressions of the causal amplitudes.

i) in covariant form

$$(4.1)\qquad \begin{pmatrix}M\\N\end{pmatrix}_{\beta\alpha}=\delta_{\beta\alpha}\left\{\begin{pmatrix}A^{(1,1)}\\B^{(1,1)}\end{pmatrix}+\frac{i}{2}\begin{pmatrix}(k+l)A^{(1,2)}\\(k'+l')B^{(1,2)}\end{pmatrix}\right\}+$$

$$+\frac{1}{2}[\tau_\beta,\tau_\alpha]\left\{\begin{pmatrix}A^{(2,1)}\\B^{(2,1)}\end{pmatrix}+\frac{i}{2}\begin{pmatrix}(k+l)A\\(k'+l')B\end{pmatrix}\right\},$$

ii) in the Breit system

$$(4.2)\qquad \begin{pmatrix}M\\N\end{pmatrix}_{\beta\alpha}=\delta_{\beta\alpha}\left\{\begin{pmatrix}C^{(1,1)}\\D^{(1,1)}\end{pmatrix}+i\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}\times\begin{pmatrix}\boldsymbol{\pi}\,C^{(1,2)}\\-\boldsymbol{\pi}'D^{(1,2)}\end{pmatrix}\right\}+$$

$$+\frac{1}{2}[\tau_\beta,\tau_\alpha]\left\{\begin{pmatrix}C^{(2,1)}\\D^{(2,1)}\end{pmatrix}+i\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}\times\begin{pmatrix}\boldsymbol{\pi}\,C^{(2,2)}\\-\boldsymbol{\pi}'D^{(2,2)}\end{pmatrix}\right\}.$$

These matrices are understood to be sandwiched by baryon spinors (Dirac spinors for (4.1) and Pauli spinors for (4.2)) and isospinors of nucleon and

(6) A. Salam: *Nucl. Phys.*, **2**, 173 (1956-1957).

K-meson. In virtue of the charge independence, amplitudes for pure isospin states, *i.e.* eigenstate of T^2, turn out to be

$$A_j^{\frac{3}{2}} = A^{(1,j)} - A^{(2,j)}, \qquad A_j^{\frac{1}{2}} = A^{(1,j)} + 2A^{(2,j)}, \tag{4.3}$$

and

$$B_j^1 = B^{(1,j)}, \qquad B_j^0 = B^{(1,j)} - 4B^{(2,j)}, \tag{4.4}$$

where the superscripts $\frac{3}{2}$, $\frac{1}{2}$, 1 and 0 indicate the eigenvalue of T. Introducing the following operators in the Breit system

$$\begin{cases} \Gamma^{(1,1)} = \dfrac{1}{4}, & \Gamma^{(1,2)} = \dfrac{-i\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}\times\boldsymbol{\pi}}{4|\boldsymbol{\Delta}\times\boldsymbol{\pi}|^2}, \\ \Gamma^{(2,1)} = -\dfrac{1}{8}[\tau_\beta, \tau_\alpha], & \Gamma^{(2,2)} = \dfrac{i\boldsymbol{\sigma}\boldsymbol{\Delta}\cdot\times\boldsymbol{\pi}}{8|\boldsymbol{\Delta}\times\boldsymbol{\pi}|^2}[\tau_\beta, \tau_\alpha], \end{cases} \tag{4.5}$$

it turns out to be

$$\begin{pmatrix} C^{(i,j)} \\ D^{(i,j)} \end{pmatrix} = \text{Trace}\, \Gamma^{(i,j)} \begin{pmatrix} M \\ N \end{pmatrix}_{\beta\alpha}, \tag{4.6}$$

where the trace is to be taken over the isospin space as well as over the spin-space. For $i = 1$, we should take $\beta = \alpha$, while for $i = 2$ it should be $\beta \neq \alpha$. Now the crossing symmetry (3.11) reads

$$C^{(i,j)}(-\omega, \Delta) = (-)^{i+j} D^{(i,j)*}(\omega, \Delta). \tag{4.7}$$

For the covariant amplitude, the crossing symmetry is exactly the same as (4.7).

Now the dispersion relations for each pair (i, j) can be written down in the following form:

$$\begin{aligned} \text{Re}\begin{pmatrix} C^{(i,j)} \\ (-)^{i+j} D^{(i,j)} \end{pmatrix}(\omega, \Delta) - \frac{P}{\pi}\int_{\omega_+}^{\infty} \frac{\text{Im}\, C^{(i,j)}(\omega', \Delta)}{\omega' \mp \omega}\, d\omega' - \\ - (-)^{i+j} \frac{P}{\pi}\int_{\omega_-}^{\infty} \frac{\text{Im}\, D^{(i,j)}(\omega', \Delta)}{\omega' \pm \omega}\, d\omega' = \frac{1}{\pi}\int_{-\omega_-}^{\omega_+} \text{Im}\, \frac{C^{(i,j)}(\omega', \Delta)}{\omega' \mp \omega}\, d\omega'. \end{aligned} \tag{4.8}$$

As a matter of fact, the r.h.s. of these equations can not be written in that way unless the analytic continuation of absorptive parts of causal amplitudes could be properly effected. However, we shall keep these forms of the r.h.s. with the understanding that the analytic continuation is assumed to be mathematically justified.

5. – Mass spectrum and unphysical region.

Though we do not touch the question of rigorous proof for analytic continuation of the absorptive parts into the unphysical region, it seems to be worth-while to get some preliminary orientations in the problem, in the hope that rigorous proof might be available in the near future and dispersion relations could be much of physical significance.

As usual, we postulate that there exists a complete set of incoming states which are eigenstates of four-momentum with non-negative values of energy $p_{n0} = E_n \geqslant 0$. We now decompose the causal amplitudes in terms of the set of intermediate states and integrate over x-space. Then the absorptive parts of the causal amplitudes turn out to be

$$\text{(5.1)} \qquad \operatorname{Im}\begin{pmatrix} C^{(i,j)} \\ D^{(i,j)} \end{pmatrix}(\omega, \Delta) = \pm \frac{(2\pi)^7}{2}\left(\frac{p_0 q_0}{M_1 M_2}\right)^{\frac{1}{2}} \Big\{ \sum_n A^{(i,j,1)}(p_n, \Delta) \cdot \\ \cdot \delta(E_n - E \mp \omega) - \sum_n A^{(i,j,2)}(p_n, \Delta)\delta(E_n - E \pm \omega) \Big\},$$

where

$$\text{(5.2)} \qquad \begin{cases} A^{(i,j,1)} = \text{Trace } \Gamma^{(i,j)} \langle \boldsymbol{\Delta}/2 \,|\, J^+(0) \,|\, \boldsymbol{p}_n = \boldsymbol{\pi}, n \rangle \langle n \,|\, j(0) \,|\, -\boldsymbol{\Delta}/2 \rangle , \\ A^{(i,j,2)} = \text{Trace } \Gamma^{(i,j)} \langle \boldsymbol{\Delta}/2 \,|\, j(0) \,|\, \boldsymbol{p}_n = -\boldsymbol{\pi}, n \rangle \langle n \,|\, J^+(0) \,|\, -\boldsymbol{\Delta}/2 \rangle . \end{cases}$$

In the derivation of (5.1), we used the *reality* of $A^{(i,j,k)}$ which can be deduced from the PT-invariance of the strong interaction.

We are now in a position to study the mass spectrum of $A^{(i,j,k)}$ in terms of selection rules, say, baryon-number conservation, strangeness conservation etc. To this end, we rewrite the δ-function in (5.1) in the following form

$$\text{(5.3)} \qquad \delta(E_n - E \mp \omega) = \frac{E_n}{E}\delta(\omega \mp \omega(\varrho^2)) ,$$

where

$$\text{(5.4)} \qquad \omega(\varrho^2) \equiv \frac{4\varrho^2 - 4E^2 - Q^2 - 2(\mu_1 + \mu_2)}{8E} .$$

ϱ denotes the mass of the intermediate state. Here we have made use of the fact $p_n = \pm \boldsymbol{\pi}$. Then the absorptive parts turn out to be

$$\text{(5.5)} \qquad \operatorname{Im}\begin{pmatrix} C^{(i,j)} \\ D^{(i,j)} \end{pmatrix} = \pm \frac{(2\pi)^7}{E}\left(\frac{p_0 q_0}{M_1 M_2}\right)^{\frac{1}{2}} \int_{\mu}^{\infty} \mathrm{d}\varrho^2 E_n(\varrho^2, \Delta) \cdot \\ \cdot \big\{ A^{(i,j,1)}(\varrho^2, \Delta)\delta(\omega \mp \omega(\varrho^2)) - A^{(i,j,2)}(\varrho^2, \Delta)\delta(\omega \pm \omega(\varrho^2)) \big\},$$

where we denoted the energy of the intermediate state by $E_n(\varrho^2, \Delta)$.

In virtue of the selection rules, we obtain the spectral representations of $\Lambda^{(i,j,1)}$ and $\Lambda^{(i,j,2)}$ in the following forms:—

$$\Lambda^{(i,j,1)}(\varrho^2, \Delta) = G_N^{(i,j)}(\Delta)\,\delta(\varrho^2 - M_1^2) + G^{(i,j,1)}(\varrho^2, \Delta)\,, \tag{5.6a}$$

where

$$G^{(i,j,1)}(\varrho^2, \Delta) = 0 \quad \text{for} \quad \varrho < M_1 + \mu_1\,,$$

and

$$\Lambda^{(i,j,2)}(\varrho, \Delta) = G_\Lambda^{(i,j)}(\Delta)\,\delta(\varrho^2 - M_\Lambda^2) + G_\Sigma^{(i,j)}(\Delta)\,\delta(\varrho^2 - M_\Sigma^2) + G^{(i,j,2)}(\varrho^2, \Delta)\,, \tag{5.6b}$$

where $G^{(i,j,2)}(\varrho^2, \Delta) = 0$ for $\varrho < M_\Lambda + \mu_1$.

The δ-functions in the spectral representation give rise to the bound-states contributions which are directly connected with the definition of the *renormalized coupling constants*. In virtue of the invariance under the inhomogeneous Lorentz transformations (including PT-invariance), selection rules and charge independence, the matrix elements of the interaction currents between initial or final baryon state and the bound-state with single baryon can be written down in the following manner:

$$\begin{cases} \langle p_n | j_\alpha(0) | p \rangle = -\dfrac{i\sqrt{2}}{(2\pi)^3}\, g_{N N\pi}\, \dfrac{M_1}{(p_{n0}p_0)^{\frac{1}{2}}}\, \bar{u}(p_n)\gamma_5 r_\alpha u(p)\,, \\ \langle q | J_\nu^\dagger(0) | p_n \rangle = \dfrac{i\sqrt{2}}{(2\pi)^3}\, g_{\Sigma N K} \left(\dfrac{M_2 M_1}{q_0 p_{n0}}\right)^{\frac{1}{2}} \bar{v}(q)\gamma_5 r_\beta u(p_n)\,, \end{cases} \tag{5.7a}$$

where $p^2 = p_n^2 = -M_1^2$, $q^2 = -M_2^2$, $(p_n - p)^2 = -\mu_1^2$, $(q - p_n)^2 = -\mu_2^2$.

$$\begin{cases} \langle q | j_\alpha^\dagger(0) | q_n \rangle = -\dfrac{\sqrt{2}}{(2\pi)^3}\, g_{\Sigma\Sigma\pi}\, \dfrac{M_\Sigma}{(q_0 q_{n0})^{\frac{1}{2}}}\, \bar{v}(q)\gamma_5 \varepsilon_{\alpha\beta\gamma} v(q_n)\,, \\ \langle q_n | J_\nu(0) | p \rangle = -i\,\dfrac{\sqrt{2}}{(2\pi)^3}\, g_{\Sigma N K} \left(\dfrac{M_2 M_1}{q_{n0} p_0}\right)^{\frac{1}{2}} \bar{v}(q)\gamma_5 r_\gamma(u)\,, \end{cases} \tag{5.7b}$$

where $q_n^2 = -M_\Sigma^2$, $(p - q_n)^2 = -\mu_2^2$, $(q_n - q)^2 = -\mu_1^2$.

$$\begin{cases} \langle q | j_\alpha^\dagger(0) | q_n \rangle = i\,\dfrac{\sqrt{2}}{(2\pi)^3}\, g_{\Sigma\Lambda\pi} \left(\dfrac{M_\Sigma M_\Lambda}{q_0 q_{n0}}\right)^{\frac{1}{2}} \bar{v}(q)\gamma_5 \delta_{\alpha\beta} v(q_n)\,, \\ \langle q_n | J_\nu(0) | p_\nu = -i\,\dfrac{\sqrt{2}}{(2\pi)^3}\, g_{\Lambda N K} \left(\dfrac{M_\Lambda M_1}{q_{n0} p_0}\right)^{\frac{1}{2}} \bar{v}(q_n)\lambda_5 u(p)\,, \end{cases} \tag{5.7c}$$

where $q_n^2 = -M_\Lambda^2$, $(M_1 \equiv M_N,\ M_2 \equiv M_\Sigma)$.

Here, we denoted the isospin variable of the Σ-hyperon in the bound state by γ, while the momenta of single nucleon bound-state and single hyperon bound-state are denoted by p_n and q_n, respectively. the K-meson is assumed to be pseudoscalar as well as the pion, in the conventional sense of the parity of the K-meson (7). In the case when the K-meson is scalar, we should only replace $i\gamma_5$ by l. The PT-invariance of the strong interaction guarantees the *reality* of all the coupling constants g, though a certain analytic consideration should be made, since in a possible rigorous proof of dispersion relations g appears to be a function of an additional variable which is introduced for the purpose of analytic continuation of the causal amplitudes into the unphysical region where p_n nor q_n are no longer real (8).

Finally, we are now in a position to write down the dispersion relations. Before doing so, we give a restriction on the momentum-transfer, in order that there may appear a gap on the unphysical region where the absorptive parts of the causal amplitude vanish. The condition for it turns out to be

$$\omega_{N\pi} > -\omega_{\Lambda\pi}\,, \qquad (\omega_{N\pi} \equiv \omega[(M_N + \mu_\pi)^2]\,,\ \text{etc.})\,,$$

or

$$\frac{Q^2}{2} < M_\Lambda(M_\Lambda + 2\mu_1) + M_N(M_N + 2\mu_1) - 2E^2 - (\mu_2^2 - \mu_1^2)\,. \tag{5.8}$$

It can be easily seen that this condition does not contradict to the requirement $Q^2 > 0$ (eq. (3.7)).

Substituting the spectral representations (5.6) into the r.h.s. of (4.8), we obtain the dispersion relations in the following form:

$$\mathrm{Re}\begin{pmatrix} C^{(i,j)} \\ (-)^{i+j} D^{(i,j)} \end{pmatrix}(\omega, \Delta) = \frac{\tilde{G}_N^{(i,j)}}{\omega_N \mp \omega} + (-)^{i+j}\frac{\tilde{G}_\Lambda^{(i,j)}}{\omega_\Lambda \pm \omega} + \tag{5.9}$$

$$+ (-)^{i+j}\frac{\tilde{G}_\Sigma^{(i,j)}}{\omega_\Sigma \pm \omega} + \frac{P}{\pi}\int_{\omega_{N\pi}}^{\infty} \frac{\mathrm{Im}\, C^{(i,j)}(\omega', \Delta)}{\omega' \mp \omega}\,\mathrm{d}\omega' + (-)^{i+j}\frac{P}{\pi}\int_{\omega_{\Lambda\pi}}^{\infty} \frac{\mathrm{Im}\, D^{(i,j)}(\omega', \Delta)}{\omega' \pm \omega}\,\mathrm{d}\omega'\,,$$

(7) G. Goebel: *Phys. Rev.*, **110**, 572 (1958).

(8) N. N. Bogoljubov, B. V. Medvedev and M. K. Polivanov: *Fortschr. d. Phys.*, **6**, 238 (1958), Appendix.

where

$$(5.10a)\qquad \tilde{G}_N^{(j,i)} = g_{\mathrm{NN}\pi} g_{\Sigma \mathrm{N}\mathrm{K}} N(M_1+M_2+2E)\left[\delta_{1j}\{-(M_1+M_2)\omega_{\mathrm{N}}+\right.$$
$$\left.+(4E)^{-1}(M_1-M_2)(4E^2+\mu_2^2-\mu_1^2-(M_1-M_2)^2)\}+\delta_{2j}\right],$$

$$(5.10b)\qquad \tilde{G}_\Lambda^{(i,j)} = g_{\Sigma\Lambda\pi} g_{\Lambda \mathrm{N}\mathrm{K}} N(M_1+M_2+2E)\delta_{1i}\left[\delta_{1j}\{-(M_1+M_2)\omega_\Lambda+\right.$$
$$+(4E)^{-1}(M_2-M_1)(4E^2+\mu_2^2-\mu_1^2-(M_1-M_2)^2)+$$
$$\left.+(2E)^{-1}(M_\Lambda-M_\Sigma)(4E^2-(M_1-M_2)^2)\}+\delta_{2j}\right],$$

$$(5.10c)\qquad \tilde{G}_\Sigma^{(i,j)} = g_{\Sigma\Sigma\pi} g_{\Sigma \mathrm{N}\mathrm{K}} N(M_1+M_2+2E)\delta_{i2}\left[\delta_{1j}\{-(M_1+M_2)\omega_\Sigma+\right.$$
$$\left.+(4E)^{-1}(M_2-M_1)(4E^2+\mu_2^2-\mu_1^2-(M_1-M_2)^2)\}+\delta_{2j}\right],$$

with

$$N \equiv \left\{4M_1M_2(M_1+p_0)(M_2+q_0)\right\}^{-\frac{1}{2}}.$$

The dispersion relations in covariant form are sometimes more convenient than those in the Breit system, *e.g.*, in the case of transforming them to c.m.s. An elementary calculation shows the following results,

$$(5.11)\qquad \mathrm{Re}\begin{pmatrix} A^{(i,j)} \\ (-)^{i+j}B^{(i,j)} \end{pmatrix}(\nu, Q^2) = g_{\mathrm{NN}\pi} g_{\Sigma\pi\mathrm{K}}\left\{(M_1-M_2)\delta_{1j}+2\delta_{2j}\right\}\frac{1}{\nu_{\mathrm{N}}\mp\nu}+$$
$$+(-)^{i+j}g_{\Sigma\Lambda\pi}\, g_{\Lambda \mathrm{N}\mathrm{K}}\,\delta_{1i}\left\{(2M_\Lambda-(M_2+M_1))\,\delta_{1j}+\delta_{2j}\right\}\frac{1}{\nu_\Lambda\pm\nu}+$$
$$+(-)^{i+j}g_{\Sigma\Sigma\pi}\, g_{\Sigma \mathrm{N}\mathrm{K}}\,\delta_{2i}\left\{(M_2-M_1)\delta_{1j}+2\delta_{2j}\right\}\frac{1}{\nu_\Sigma\pm\nu}+$$
$$+\frac{P}{\pi}\int_{\nu_{\mathrm{N}\pi}}^{\infty}\frac{\mathrm{Im}\,A^{(i,j)}(\nu', Q^2)}{\nu'\mp\nu}\,\mathrm{d}\nu'+(-)^{i+j}\frac{P}{\pi}\int_{\nu_{\pi\mathrm{N}}}^{\infty}\frac{\mathrm{Im}\,B^{(i,j)}(\nu', Q^2)}{\nu'\pm\nu}\,\mathrm{d}\nu'.$$

The values of ω_{N}, ω_Λ, etc. (correspondingly ν_{N}, ν_Λ, etc.) are given by Eq. (5.4).

So far we have considered only the case of Σ-hyperon production by pion-nucleon collision. It is now rather straightforward to obtain the dispersion relations for the Λ-hyperon production. The only difference comes about from the isoscalar nature of the Λ-hyperon, while the Σ-hyperon is an isovector. Through the invariance consideration we can express the causal amplitudes for the Λ-production process and its crossing process in the following covariant

forms:

$$M_\alpha(\nu, Q^2) = \tau_\alpha \left[A^{(1)}(\nu, Q^2) + \frac{i}{2}(k + l)\, A^{(2)}(\nu, Q^{(2)}) \right], \tag{5.12a}$$

$$N_\alpha(\nu, Q^2) = \tau_\alpha \left[B^{(1)}(\nu, Q^2) + \frac{i}{2}(k' + l')\, B^{(2)}(\nu, Q^2) \right], \tag{5.12b}$$

$(M_1 \equiv M_N,\ M_2 \equiv M_\Lambda)$

where α stands for the isospin variable of the incoming pion. Then the crossing symmetry now turns out to be

$$A^{(j)}(-\nu, Q^2) = (-)^j B^{(j)*}(\nu, Q^2)\,. \tag{5.13}$$

Thus the dispersion relations for Λ-production reads as follows:

$$\begin{aligned} \text{(5.14)} \qquad \operatorname{Re} \begin{pmatrix} A^{(j)} \\ (-)^j B^{(j)} \end{pmatrix} (\nu, Q^2) = {} & g_{N\Sigma\pi}\, g_{\Lambda N K} \left\{ (M_N - M_\Lambda)\,\delta_{1j} + 2\delta_{2j} \right\} \frac{1}{\nu_N \mp \nu} + \\ & + (-)^j g_{\Sigma\Lambda\pi}\, g_{\Sigma N K} \left\{ (2M_\Sigma - M_N - M_\Lambda)\,\delta_{1j} + 2\delta_{2j} \right\} \frac{1}{\nu_\Sigma \pm \nu} + \\ & + \frac{P}{\pi} \int_{\nu_{N\pi}}^{\infty} \frac{\operatorname{Im} A^{(j)}(\nu', Q^2)}{\nu' \mp \nu}\, \mathrm{d}\nu' + (-)^j \frac{P}{\pi} \int_{\nu_{\Lambda\pi}}^{\infty} \frac{\operatorname{Im} B^{(j)}(\nu', Q^2)}{\nu' \pm \nu}\, \mathrm{d}\nu'\,. \end{aligned}$$

6. – Concluding remarks.

So far, we have derived dispersion relations for the associated production only in a formal way without rigorous proof. There are some indications that dispersion relations could not be proved for the processes with higher threshold energy which allows the existence of competing channels with lower threshold energy, so that the unphysical continuum does not disappear even for zero momentum-transfer ([1]). However, the possibility of proving dispersion relations is not yet excluded until we find integral representations of causal commutators which include all the available information contained in the basic axioms of the relativistic quantum field theory, *e.g.*, the unitarity of the S-matrix which has not yet been effectively used in the proof of dispersion relations.

Even if the dispersion relations could be proved in a rigorous way, its practical applications would involve very complicated calculations, since there appear various competing processes even at the threshold of the relevant process so that the use of the unitarity of the S-matrix which would connect the

scattering amplitudes for various competing channels would become extremely involved, besides the awkward algebra due to the inelastic nature of the relevant processes. Nevertheless, it will turn out to be rather immediate to obtain dispersion relations in static approximation and incorporate it with the unitarity of S-matrix with an additional assumption that higher order terms in K-meson coupling can be neglected in comparison with that of pion-baryon coupling (9).

(9) D. AMATI and B. VITALE: *Nuovo Cimento*, **9**, 895 (1958).

RIASSUNTO (*)

Si presenta una derivazione formale delle relazioni di dispersione relativistiche per la produzione associata di particelle strane nelle collisioni pione-nucleone, senza tuttavia tentare di dare una prova rigorosa della loro derivazione in base alla continuazione analitica delle ampiezze nella regione non fisica. Si studia la struttura delle matrici esprimenti le ampiezze casuali nello spazio dello spin e dell'isospin. Si ottengono relazioni di dispersione sia in forma covariante che nel sistema di Breit, separatamente per processi con spin-flip e senza spin-flip e con isospin-flip e non isospin-flip della produzione associata.

(*) *Traduzione a cura della Redazione.*

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Proprietà focalizzanti del campo di Biot-Savart.

E. R. Caianiello e G. Gatti

Istituto di Fisica Teorica dell'Università - Napoli
Scuola di Perfezionamento in Fisica Teorica e Nucleare del C.N.R.N. - Napoli

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Riassunto. — Vengono studiate le proprietà focalizzanti dei campi magnetici prodotti da correnti rettilinee di altissima intensità. I risultati ottenuti sembrano indicare che tali campi possano essere utilmente impiegati nella costruzione di acceleratori di particelle.

1. – Introduzione.

Negli ultimi anni sono state spesso usate in macchine di ricerca scientifica correnti impulsive di notevole intensità, dell'ordine di 10^6 A ed oltre.

Esempi del genere si hanno nelle macchine per ricerche sui plasmi, come pure nel sincrotrone di Canberra [1], in cui il campo magnetico è ottenuto, senza l'impiego di materiale ferromagnetico, con una corrente di 10^7 A, erogata da una macchina omopolare.

La possibilità di ottenere forti campi magnetici ha suggerito l'idea di studiare la focalizzazione di un fascio di particelle cariche nel caso semplice e tuttavia sufficientemente istruttivo del campo di Biot-Savart (prodotto da una scarica attraverso un conduttore metallico cilindrico o da una colonna di ioni).

Fatta l'ipotesi adiabatica, per cui l'intensità della corrente non varia sensibilmente nel tempo impiegato dalle particelle a descrivere un giro, dallo studio delle equazioni del moto si ricava la possibilità di orbite interne ad una zona toroidale per particelle iniettate normalmente al conduttore cilindrico.

Le conclusioni di questo studio sembrano favorevoli alla possibilità di realizzare macchine acceleratrici con tali metodi, che potranno divenire di interesse concreto in un futuro non lontano. Si accennerà, negli ultimi due paragrafi, a qualche possibile spunto costruttivo.

2. - Moto di particella carica in un campo di Biot-Savart.

Avendo fatta l'ipotesi adiabatica, studieremo il moto di una carica elettrica q in un campo costante di Biot-Savart ($H =$ in unità M.K.S.).

Le equazioni del moto, espresse in coordinate cilindriche in cui l'asse Z coincide con l'asse del conduttore, sono

$$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = v^2 , \tag{1}$$

$$r^2\dot{\theta}^2 = r_0^2\dot{\theta}_0 = A , \tag{2}$$

$$\dot{z} = \dot{z}_0 + K \log \frac{r}{r_0} , \tag{3}$$

dove

$$K = \frac{\mu_0 q i}{2\pi m} . \tag{4}$$

Le (1), (2), (3) valgono ovviamente anche nel caso relativistico; m è infatti la massa relativistica della carica q.

Dalle (1), (2), (3) si ricavano le

$$\dot{r}^2 + \frac{A^2}{r^2} + \left(\dot{z}_0 + K \log \frac{r}{r_0}\right)^2 = v_0^2 , \tag{5}$$

$$\ddot{r} = \frac{r_0^2\dot{\theta}_0^2}{r^3} - \frac{K}{r}\left(\dot{z}_0 + K \log \frac{r}{r_0}\right) , \tag{6}$$

di cui ci serviremo per lo studio della focalizzazione delle particelle. Consideriamo ora il caso di una particella iniettata perpendicolarmente al conduttore con notevole velocità iniziale. Le condizioni iniziali di questa particella sono espresse dalle

$$\dot{r}_0 = \dot{z}_0 = 0 , \tag{7}$$

$$\dot{\theta}_0 \neq 0 , \tag{8}$$

$$\theta_0 = z_0 = 0 , \tag{9}$$

$$v_0 = r_0\dot{\theta}_0 . \tag{10}$$

Con il cambiamento di variabile da t a θ effettuato tramite la

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} , \tag{11}$$

e le posizioni

$$y = \frac{r_0}{r}, \tag{12}$$

$$\lambda = \frac{K}{v_0} = \frac{\mu_0 q i}{2\pi m v_0}, \tag{13}$$

la (5) può essere trasformata tramite le (7)-(10) nella

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 + y^2 + \lambda^2 \log^2 y = 1. \tag{14}$$

Dalla (14) si ricava la

$$y^2 + \lambda^2 \log^2 y \leqslant 1, \tag{15}$$

che porta alla

$$y \leqslant 1, \tag{16}$$

nonchè alla diseguaglianza

$$\lambda^2 \log^2 y < 1, \tag{17}$$

da cui segue

$$y > \exp\left[-\frac{1}{|\lambda|}\right]. \tag{18}$$

In definitiva, le (16) e (18) equivalgono alla

$$r_0 \leqslant r < r_0 \exp\left[\frac{1}{|\lambda|}\right], \tag{19}$$

da cui si nota che per $|\lambda|$ piuttosto grande r non si allontana troppo dal valore iniziale r_0.

Dall'equazione (14) si ricavano i valori per cui y è minima e massima nell'intervallo $(0, 1)$. Tali valori sono due: $y = 1$ e $y = \overline{y}$, le radici dell'equazione

$$y^2 + \lambda^2 \log^2 y = 1. \tag{20}$$

Il valore della $\mathrm{d}^2y/\mathrm{d}\theta^2$ è ottenuto derivando la (14) rispetto a θ

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2} = -y - \frac{\lambda^2}{y} \log y. \tag{21}$$

Tale derivata seconda ha valore negativo per $y = 1$; ciò significa che quel punto è un massimo.

Invece per $y = \bar{y}$ la (21) assume un valore positivo, come si può dimostrare sostituendo nel secondo membro il valore di λ^2 dato dalla

$$\lambda^2 = \frac{1 - \bar{y}^2}{\log^2 \bar{y}} \,. \tag{22}$$

Con questa posizione tutto si riduce a dimostrare la relazione

$$-\bar{y}^2 - \frac{1 - \bar{y}^2}{\log \bar{y}} > 0 \,, \tag{23}$$

e qui ponendo

$$\bar{y} = \exp[-x] \,, \tag{24}$$

(con x compreso tra 0 e ∞, perchè $\bar{y}$ resti compreso tra 0 ed 1) si ottiene

$$\exp[2x] > x + 1 \,, \tag{25}$$

che è sicuramente verificata.

Quindi $\bar{y}$ è un minimo: ciò assicura che r assume un valore massimo pari a $r_{max} = r_0/\bar{y}$, per poi tornare indietro.

Resta però da verificare che la variabile y passa dal valore 1 al valore $\bar{y}$ in un tempo finito, il che comporta la convergenza dell'integrale

$$\Delta\theta = 2\int_{\bar{y}}^{1} \frac{\mathrm{d}y}{\sqrt{1 - y^2 - \lambda^2 \log^2 y}} \,. \tag{26}$$

L'integrale (26) converge perchè la funzione

$$F(y) = 1 - y^2 - \lambda^2 \log^2 y \,, \tag{27}$$

tale che

$$\frac{\mathrm{d}F}{\mathrm{d}y} = 2\,\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2} \,,$$

ha come radici semplici 1 ed $\bar{y}$.

Inoltre nel variare di y da 1 a $\bar{y}$ e viceversa si ha anche uno spostamento Δz espresso da

$$\Delta z = -2\lambda r_0 \int_{\bar{y}}^{1} \frac{\log y \,\mathrm{d}y}{y^2 \sqrt{1 - y^2 - \lambda^2 \log^2 y}} \,. \tag{28}$$

3. – Stabilità del moto.

Occorre studiare se particelle con condizioni iniziali diverse, per quantità infinitesime, da quelle delle particelle di riferimento, si allontanino indefinitamente da essa o no.

Consideriamo perciò una particella che inizia il suo moto dalla posizione r_0, con $\theta_0 = z_0 = 0$ ed $\dot{r}_0 = \dot{\varrho}_0$ e $\dot{z}_0 = \dot{\zeta}_0$ quantità infinitesime del primo ordine. Poniamo

$$\begin{cases} r = R + \varrho\,, \\ \theta = \varphi + \psi\,, \\ z = Z + \zeta\,, \end{cases} \tag{29}$$

dove R, Z, φ si riferiscono al moto della particolare particella della sezione precedente.

Sostituendo nella (6) ed eliminando dalle funzioni sviluppate in serie i termini di ordine superiore al primo in ϱ, ζ e ψ otteniamo

$$\ddot{\varrho} + \omega^2 \varrho = -\frac{K}{R}\dot{\zeta}_0\,, \tag{30}$$

dove

$$\omega = v_0 \sqrt{\frac{3r_0^2}{R^4} + \frac{\lambda^2}{R^2}\left(1 - \log\frac{R}{r_0}\right)}\,. \tag{31}$$

Detto $\overline{\omega}$ il valore medio di ω la soluzione della (30) è del tipo

$$\varrho = \frac{\dot{\varrho}_0}{\omega}\sin\overline{\omega}t + \frac{K_0\dot{\zeta}}{R\overline{\omega}^2}(\cos\overline{\omega}t - 1)\,, \tag{32}$$

dove si ha contemporaneamente $\sin\overline{\omega}r = 0$; $\cos\omega\overline{r} - 1 = 0$ e quindi la stabilità radiale è assicurata.

Procedendo analogamente per la variazione angolare otteniamo

$$\psi \simeq \frac{2v_0 r_0}{R^3}\frac{\varrho_0}{\omega^2}\cos\overline{\omega}t\,. \tag{33}$$

Nello studio della stabilità verticale, invece, perveniamo alla

$$\zeta \simeq \dot{\zeta}_0 t - \frac{K\dot{\varrho}_0}{R\overline{\omega}^2}\cos\overline{\omega}t \simeq \zeta_0 t\,. \tag{34}$$

Quindi nella direzione dell'asse z non si ha stabilità e si deve cercare di ottenerla con dei campi aggiunti.

4. – Sulla possibilità di utilizzare un campo di Biot-Savart in nuovi tipi di macchine acceleratrici.

Nelle tre Sezioni precedenti abbiamo studiato il moto di un fascio di particelle, iniettate ortogonalmente ad un conduttore rettilineo indefinito percorso da corrente continua.

Abbiamo visto che le particelle percorrono orbite composte da un moto nel piano (x, y), compreso fra i cerchi di raggio r_0 ed $r_{max} = r^0/\overline{y}$, e da uno spostamento lungo l'asse z, espresso dal Δz della formula (28) e da $\dot{\zeta} = \zeta_0 t$.

Quindi un campo di Biot-Savart può essere utilizzato per far descrivere ad un fascetto di particelle un'orbita chiusa, purchè con campi aggiunti si riesca a impedirne una troppo grande elongazione in direzione z.

Come dispositivo accelerante potrebbero essere adoperati due « D » simili a quelli di un ciclotrone.

Però la possibilità di realizzare una macchina acceleratrice di questo tipo è legata a quella di realizzare un opportuno coefficiente λ, che è in diretta relazione coi valori di r_0 e $r_{max} = r_0/\overline{y}$ tramite la

$$\lambda = \frac{1 - \overline{y}^2}{\log^2 \overline{y}} . \tag{35}$$

Alcuni valori di λ per diversi $\overline{y}$ sono qui riportati:

$\overline{y}$	λ	$\overline{y}$	λ
0.80	3.2	0.4	1
0.7	2.18	0.3	0.75
0.5	1.25	0.2	0.61

Valori così elevati di λ si ottengono solo con forti correnti, come è espresso dalla

$$\lambda = 6 \cdot 10^{-5} \frac{i}{W_{(MeV)}} , \tag{36}$$

ricavata dalla (13) tenendo conto che $v_0 \simeq c$, ed esprimendo l'energia totale della particella, $W = mc^2$, in MeV e l'intensità della corrente in A.

Ad esempio, per ottenere dei valori di λ dell'ordine di 0.8 occorrono intensità di correnti dell'ordine di 10^5 A per energie di 10 MeV.

Non è possibile realizzare correnti continue di tale intensità, ma solo correnti impulsive.

Una volta nota la legge di variazione di $i(t)$ in funzione del tempo, bisogna incrementare l'energia della particella W in modo che il rapporto $\lambda = 6.10^{-5}(i/W)$ resti costante, con conseguente costanza del valore di $\overline{y} = r_0/r_{max}$ e dei cerchi massimo e minimo.

I problemi che si presentano per la realizzazione di questa forte corrente impulsiva sono notevoli e richiedono studi particolari che fuoriescono dagli scopi del presente lavoro. Riteniamo tuttavia utile accennare, per concludere, a qualche possibile soluzione concreta.

Potrebbe essere conveniente usare un circuito a tubo coassiale dove un cilindro interno C, che è il nostro conduttore rettilineo, è unito alla base ad un cilindro esterno cavo C', ad esso coassiale, che serve per il ritorno della corrente.

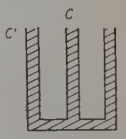

Fig. 1.

In questa maniera si otterrebbe un circuito con bassa autoinduzione, in cui sarebbe facile controllare le forti reazioni elettrodinamiche.

Inoltre, realizzando il circuito con sbarrette sottili, isolate ed intrecciate fra di loro, si ridurrebbe di molto l'effetto pellicolare e quindi la resistenza ohmica, responsabile del riscaldamento per effetto joule.

L'elongazione lungo l'asse z potrebbe essere corretta con il campo elettrostatico stabilito tra due dischi paralleli ed ortogonali al cilindro.

5. – Studio della soluzione elicoidale dell'equazione del moto.

Dalle (1), (2), (3), (6) risulta che è possibile una soluzione per cui r sia costante. Per essa si ha contemporaneamente

$$\dot{\theta} = \dot{\theta}_0 \,, \tag{37}$$

$$r = r_0 \,, \tag{38}$$

e poichè deve essere $\ddot{r} = 0$ la z è vincolata a soddisfare la

$$\dot{z}_0 = \frac{r_0^2 \dot{\theta}_0^2}{K} \,. \tag{39}$$

Tale valore della componente $\dot{z}$ viene mantenuto costante durante tutto il moto, come risulta dalla (3).

Si realizza così un'orbita elicoidale: tale possibilità può essere studiata per un dispositivo acceleratore di tipo diverso, costituito da due o più conduttori disposti in croce.

La particella avanza elicoidalmente lungo uno di essi, per poi avvitarsi lungo l'altro, una volta cessata l'influenza del primo.

Questo dispositivo presenta però delle difficoltà tecniche di gran lunga superiori a quelle del tubo coassiale, costituite dalle notevoli azioni elettrodinamiche che si generano.

È da notare, però, che questo dispositivo non richiede campi aggiunti che contengano l'elongazione lungo l'asse z.

SUMMARY

The focusing properties of magnetic fields produced by rectilinear currents of very high intensity are investigated. The results seem to indicate that such fields might prove of practical use in the design of particle accelerators.

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Production of the Magnetic Field of the Crab Nebula (*).

L. Marshall (**)

The Enrico Fermi Institute for Nuclear Studies
The University of Chicago - Chicago, Ill.

(ricevuto il 14 Febbraio 1959)

Summary. — The magnetic field of the Crab Nebula is so large that if the original pre-super nova star had contained it, it must have been on the verge of instability. It is suggested here that this magnetic field did not exist before the explosion which produced the Crab, but was created from the kinetic energy of the expanding matter. To account for the observed magnetic field of $\sim 10^{-3}$ G a machinery is proposed which derives from the existence of shock waves in the exploding shell, from the existence of charge separation in the shock fronts, and from the formation of irregularities in the surfaces where charge separations exist, at some time during their propagation from regions of very high density in the star interior out into the very low galactic density. The magnetic field of 10^{-3} G, now believed to exist in the Crab, follows from the present theory, if one assumes the experimental value of $10^3/cm^3$ for the density in the filaments.

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1. – The existence in the Crab Nebula of a magnetic field density of 10^{-3} G (Oort and Walraven [1]) provides an interesting problem in that such a field, if compressed from its present volume into the volume of the original pre-super nova, would have a magnetic energy content so great as to make one doubt the stability of the star (Chandrasekhar [2]). In the present paper,

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(**) Presently on leave of absence to Brookhaven National Laboratory, Upton, New York.

(1) J. H. Oort and Th. Walraven: *Bull. Am. N.*, **12**, no. 462, 285 (1956).

(2) S. Chandrasekhar: *MNRAS*, **113**, 667 (1953). See also footnote p. 304 in Oort and Walraven (1).

it is proposed that this magnetic field was not there originally, but on the contrary was created out of the kinetic energy of the expanding super nova.

The method proposed here for conversion of kinetic to magnetic energy depends on *a*) the production of shock waves in the exploding supernova, *b*) the natural existence of charge separation in shock fronts, and *c*) the existence of fine structure geometric irregularities in the surface of an exploding star. Briefly this mechanism depends on the following argument. Although the charge separation in moving hot material acts like an electric current, yet for a spherically symmetric geometry, owing to its symmetry, all accompanying magnetic fields cancel. However, if there are fine structure irregularities in the surface where charge separation is maintained, the magnetic fields no longer cancel to zero, but instead grow in proportion to the increasing charge separation, which is a consequence of the decreasing density, as the expanding material emerges into interstellar regions.

The first of these statements, the hypothesis of the production of shock waves in the explosion of a supernova, we take as self-evident. The second statement, that of the common occurrence of charge separation in all shock fronts in which the temperature is high enough to ionize electrons, may be understood as a consequence of the light mass of the electron which consequently has greater velocity and goes out ahead of a proton of the same energy. The phenomenon of charge separation in shock waves has been discussed qualitatively by KANTROWITZ [3]. Furthermore the difference in electric potential across a shock front has been demonstrated and measured for a shock propagating in argon by PETSCHEK and BYRON [4].

The third point on which this mechanism depends is that of the inability of exploding matter to maintain a geometry which is a smooth surfaced expanding sphere. This is a classical case of instability and has been discussed by LAMB [5] and by TAYLOR [5]. It is a reasonable supposition that such deformations of the emerging matter are features of every exploding star and must produce a wide variety of jets, spurts, and erratic excrescences of the outwardly expanding material.

2. – Let L be the dimension of such an excrescence, and let D be the distance of charge separation in the shock front which borders the excrescence. Let us assume that the electron sheath propagates in such a way that it is not destroyed by plasma oscillations because the mean free path is large com-

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[3] A. KANTROWITZ: *Experiments on the radiation and ionization produced by strong shock waves*, in *Gas Dynamics of Cosmic Clouds, A Symposium* (New York, 1955).

[4] H. PETSCHEK and S. BYRON: *Ann. of Phys.*, **1**, 270 (1957).

[5] See H. LAMB: *Hydrodynamics*, 6th ed. (1932), p. 374; G. I. TAYLOR: *Proc. Roy. Soc.*, **201**, 192 (1950); and D. J. LEWIS: *Proc. Roy. Soc.*, **202**, 81 (1950).

pared with the distance of charge separation. The equation of forces is written

$$Nm\ddot{x} = Ne\left(E + \frac{1}{c} v \times H\right) + \operatorname{grad} P, \tag{1}$$

where P is pressure, E is the electric field produced by charge separation, and H is the magnetic field produced by motion of E with shock velocity v.

Consequently H is smaller than E by the ratio v/c and its reaction on E may be neglected for purposes of estimating E. The time average of the term which describes plasma oscillations, $Nm\ddot{x}$, where N is electron density and $m\ddot{x}$ is the acceleration, is zero. Then it follows that

$$Ne\boldsymbol{E} = \nabla P, \tag{2}$$

from which the potential difference across the distance D of charge separation is found to be

$$ED = \frac{kT}{e} \int \frac{\nabla P}{P} = \frac{kT}{e} \ln \frac{P_2}{P_1}. \tag{3}$$

Using the value $E = 4\pi NDe$ for the electric field in the same region, we find

$$D = \left[\frac{kT}{4\pi Ne^2} \ln \frac{P_2}{P_1}\right]^{\frac{1}{2}}. \tag{4}$$

In the rest system a magnetic field H is produced by the moving electric field, such that its lines of force form rings around the material excrescences. H is given by

$$\boldsymbol{H} = \nabla \times \boldsymbol{A} = \nabla \times \int i \frac{\mathrm{d}\boldsymbol{s}}{R}.$$

Here the increment of current $\mathrm{d}s$ is equal to D, and the distance R from the center of the excrescence to the point where H is observed is approximately L. It follows that contributions to H from various parts of the moving shock front are weighted by D/L. Consequently for purposes of making a magnetic field, the most effective irregularities in the expanding matter are those for which L is not too large compared with D; for example, $L \leqslant 100\, D$.

The magnetic field may be estimated from a line integral around a moving excrescence

$$\int \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = 4\pi i,$$

where i, the current equivalent to the moving electric field, is given by

$$i = \frac{v}{c} NL^2 e \ .$$

Then

$$(6) \qquad \begin{cases} H 2\pi L = 4\pi N e L^2 \dfrac{v}{c}, \\ H = 2 N e L \dfrac{v}{c}. \end{cases}$$

Assuming $L \sim 10\, D$ and putting (4) into (6) we obtain

$$H = 20 \frac{v}{c} \left[\frac{NkT}{4\pi} \ln \frac{P_2}{P_1} \right]^{\frac{1}{2}}.$$

This field is produced in the rest system, *i.e.* the system in which the center of the Crab nebula is at rest, by a shock front moving with relative velocity v.

Let the ln term, which is the sum of the terms $\ln T_e'/T_e$ and $\ln N_e'/N_e$, where T_e and N_e are electron temperature and density, be about 10, and the proton velocity be $1.5 \cdot 10^8$ cm/s. It follows that $H = 1.3 \cdot 10^{-5}\, N_e^{\frac{1}{2}}$ and that, therefore, a magnetic field of $4 \cdot 10^{-4}$ G is produced in a region having an electron density of $10^3/\mathrm{cm}^3$. We may compare these numbers to the experimental values. Namely, from his spectroscopic measurements of light emitted from filaments, OSTERBROCK (6) has estimated the electron density to be $10^3/\mathrm{cm}^3$, and the magnetic field has been estimated as $10^{-4} \leqslant H \leqslant 10^{-3}$ G by OORT and WALRAVEN (1).

The corresponding value of D is computed from Eq. (4) to be 10^4 cm, and consequently $L \geqslant 10^5$ cm.

3. – Let us consider the direction of the polarization observed at the earth for synchrotron radiation emitted by electrons in the shock front. If the explosion took place in a spherically symmetrical way on the macroscopic scale, the fields produced by the machinery described here would have ring shaped lines of force lying on the spherically exploding shell. High energy electrons bending around the lines of force emit a narrow cone of synchrotron radiation which irradiates the earth only when the electron's velocity points toward the earth. The electric vector of the radiation is perpendicular to the velocity and to the line of force. Consequently, for those field loops near the

(6) D. E. OSTERBROCK: *Astron. Journ. Pacif.*, **69**, no. 408, 227 (1957).

limb of the exploding shell, as seen from the earth, observation of synchrotron radiation should show the electric vector of the radiation approximately along a radius of the sphere. This is not the case in the Crab, there being instead dark bays on the east and west limbs of the expanding shell. Furthermore, according to this model, one expects to see the polarization in the central region on the surface of the exploding sphere go to zero, because in this case, one looks at magnetic field loops lying in a plane perpendicular to the line of sight, and consequently the polarized light coming from any complete loop takes all directions and averages to zero.

However, suppose the pre-super nova had a dipole magnetic field of the order of 1 G, like that of the sun, and directed roughly perpendicular to the line of sight, for example directed north and south. Then the growth of material excrescences in shock fronts crossing the original magnetic field direction would be inhibited, but those in shock fronts moving toward the original magnetic poles would grow as before. As a result, magnetic field loops produced by this machinery would be preferentially oriented with their planes perpendicular to the north-south direction. In this case, synchrotron radiation reaching the earth would have an electric vector mainly north-south oriented, and the apparent magnetic field that one would deduce would seem to have roughly parallel lines of force. In fact, it would seem to have great homogeneity.

The field of the Crab seems to be moderately well described by this latter situation. Namely, the electric vectors of synchrotron radiation from various parts show an overall north-south predominance, and a rather high degree of homogeneity.

However, the Crab has a more complicated structure, in that apparently the inner region is composed of a magnetic field of $\sim 10^{-3}$ G containing a very low density of matter. The model described here suggests that after the expanding front has passed by, the magnetic field contains enough electrons scattered into it during its creation that it cannot die away, but instead is maintained by the particles contained in it. That is, the low density inner region of the Crab is a region of fossil magnetic field preserved by a relatively small number of electrons and protons which were stored there while the field was being made. This requires that the magnetic energy density $H^2/8\pi = 4 \cdot 10^{-8}$ erg/cm^3 shall equal the kinetic energy density Qcp where Q is the number of electrons/cm^3 of average momentum cp. If cp is 10^{11} eV, as has been suggested by many authors, then a value $Q \sim 3 \cdot 10^{-7}$ electrons/cm^3 is an electron density sufficient to maintain the fossil magnetic fields inside the expanding shell.

It may be remarked that in order for synchrotron radiation of wave length λ to be polarized, the minimum size L of regions of homogeneous magnetic field must be larger than λ. In particular, the Crab emits radio waves which

have been interpreted as synchrotron radiation (OORT and WALRAVEN [1]), but no polarization has yet been observed at radio frequencies. According to the present model, with $L \geqslant D \sim 10^4$ cm, radiation of wave length $\geqslant 10^4$ cm ($\geqslant 3$ MHz) can be polarized.

The mechanism described above for production of the magnetic fields of the Crab out of the kinetic energy still available from the explosion of the original super-nova provides an automatic explanation for the near equivalence of magnetic field energy and gravitational energy density. According to this model, no magnetic field, no synchrotron radiation, and very few fast electrons are expected to be found outside of the expanding shell produced in the explosion of the super nova.

RIASSUNTO (*)

Il campo magnetico della nebulosa del Granchio è così vasto che se la pre-supernova originale l'avesse contenuto avrebbe dovuto trovarsi al limite dell'instabilità. Si suggerisce qui che questo campo magnetico non esisteva prima dell'esplosione che produsse il Granchio, ma fu invece creato dall'energia della materia in espansione. Per render conto del campo magnetico creato di $\sim 10^{-3}$ G si propone un meccanismo derivante dall'esistenza di onde d'urto nell'esplosione della stella, dall'esistenza di separazione di carica nei fronti d'urto, e dalla formazione di irregolarità sulle superfici dove esistono separazioni di carica in qualche tempo durante la loro propagazione da regioni di altissima densità dell'interno della stella a regioni di bassissima densità galattica. Il campo magnetico di 10^{-3} G, che presentemente si ritiene esistere nel Granchio, consegue dalla teoria esposta se per la densità dei filamenti si assume il valore sperimentale di 10^3/cm^3.

(*) *Traduzione a cura della Redazione.*

On the Transverse Momenta of Secondaries in Cosmic-Ray Jets.

E. M. FRIEDLÄNDER

Cosmic Ray Laboratory, Institute of Atomic Physics - Bucharest

(ricevuto il 19 Gennaio 1959)

1. – Recent investigations have shown that the transverse momentum p_T of secondaries from high-energy cosmic ray jets fluctuates about a relatively low mean of the order of the pion rest mass μ [1]. The distribution function of $\log p_T$ has been shown to be of gaussian form [2]. This confirms previous observations (*e.g.* [3, 4]) of low average c.m.s. energies of the secondaries.

An important consequence of these facts is that the conventional methods used for the estimation of the c.m.s. Lorentz factor γ_c must lead — in the high energy range — to a serious overestimation of this quantity and hence of the primary energy E. Indeed, of the two assumptions underlying these methods, *viz.* forward-backward symmetry in the c.m.s. and equality of the c.m.s. velocity β_c and of the particle velocity v in the c.m.s., the second one is no more valid. It is easy to show that the correct measure for the departure from the condition $v=\beta_c$ is given just by the transverse momentum p_T. We shall do this first on the example of the quantal method [5].

Consider two symmetrical particles emitted in the lab. system at angles θ_f and θ_{1-f}, such that the cones delimited by these angles include fractions f and $(1-f)$ of the produced particles. Define now a quantity

$$\gamma_f^2 \equiv \operatorname{ctg} \theta_f \operatorname{ctg} \theta_{1-f} . \tag{1}$$

If v_f and φ_f are the velocity and angle in the c.m.s. of particle «f», and

$$m_f \equiv \frac{\beta_c}{v_f} . \tag{2}$$

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[1] Z. KOBA: *Progr. Theor. Phys.*, **15**, 461 (1956).
[2] G. B. ŽDANOV: *Žu. Éksp. Theor. Fiz.*, **34**, 856 (1958).
[3] D. HASKIN, R. GLASSER, M. SCHEIN and J. LORD: *Phys. Rev.*, **99**, 1555 (1955).
[4] V. HOPPER, S. BISWAS and J. DARBY: *Phys. Rev.*, **80**, 970 (1950).
[5] M. F. KAPLON, B. PETERS and H. BRADT: *Helv. Phys. Acta*, **23**, 24 (1950).

the relativistic angle transform yields:

$$\overline{\gamma_f^2} = \overline{\gamma_c^2}\left(1 + \frac{\overline{m_f^2} - 1}{\overline{\sin^2 \psi_f}}\right). \tag{3}$$

It is usually assumed that at high energies ($\gamma_c \gg 1$) the average c.m.s. energy of the secondaries is so high, that $v_f \approx 1$ and hence $m_f \approx 1$. Then γ_f is independent of f. ($\gamma_f \approx \gamma_c$) and it represents the quantal estimate for γ_c. Now, from eq. (3) one sees at once that small departures of m_f from unity may become of importance if $\sin \psi_f$ is small enough. If g_f is the Lorentz factor corresponding to v_f, ($g_f = (1 - v_f^2)^{-\frac{1}{2}}$) and π_f the transverse momentum expressed in units of the pion rest mass:

$$\pi_f \equiv (g_f^2 - 1)^{\frac{1}{2}} \sin \psi_f , \tag{4}$$

we obtain from (3) and (4)

$$\gamma_f^2 = \gamma_c^2 \left(1 + \frac{1 - (g_f/\gamma_c)^2}{\pi_f^2}\right). \tag{5}$$

Eq. (5) shows clearly that the usual method of γ_c-estimation ($\gamma_c \approx \gamma_f$) is valid *only at low energies* ($\gamma_c \sim g_f$) while at high energies ($\gamma_c \gg g_f$) we have

$$\gamma_f^2 \approx \gamma_c^2 (1 + \pi_f^{-2}) . \tag{6}$$

A similar argument applies to the Castagnoli formula [6]:

$$\ln \gamma_c = \overline{\ln \operatorname{ctg} \theta} , \tag{7}$$

which has to be replaced by

$$\ln \gamma_c = \overline{\ln \operatorname{ctg} \theta} - \frac{1}{2N} \sum_i \ln \left(1 + \frac{1 - (g_i/\gamma_c)^2}{\pi_i^2}\right), \tag{8}$$

where N is the number of shower particles (see Fig. 1).

We suggest that γ_c should be estimated by means of the r.m.s. value Γ of γ_f:

$$\Gamma = \left[\frac{2}{N} \sum_f \gamma_f^2\right]^{\frac{1}{2}}. \tag{9}$$

The expectation value of Γ can be computed using the gaussian distribution of the quantity (cfr. [2])

$$x \equiv \ln \pi_f . \tag{10}$$

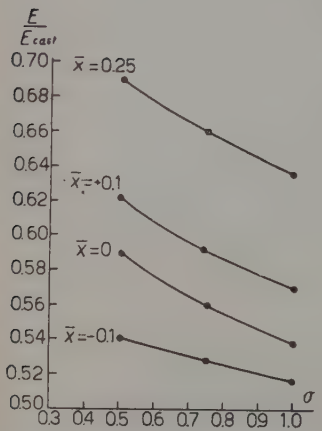

Fig. 1. – The average ratio of the true primary energy E to the Castagnoli estimate E_{Cast} obtained by averaging (numerically) eq. (8) over x for different $\bar{x}$ and σ.

(6) C. Castagnoli, G. Cortini, C. Franzinetti, A. Manfredini and D. Moreno: *Nuovo Cimento*, **10**, 1539 (1953).

If $\bar{x}$ and σ^2 are the mean and the variance of this distribution, Γ is given by

$$\Gamma^2 = \gamma_c^2 \left[1 + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left\{ -\left(\frac{(x-\bar{x})^2}{2\sigma^2} + 2x \right) \right\} \frac{\mathrm{d}x}{\sigma\sqrt{2}} \right]. \tag{11}$$

The integration is readily performed and yields

$$\gamma_c = \Gamma(1 + \exp\left[2(\sigma^2 - \bar{x})\right])^{-\frac{1}{2}}. \tag{12}$$

For each individual jet σ^2 and $\bar{x}$ can be estimated by measuring the transverse momenta (which are Lorentz invariant) in the lab. system. Thus we obtain an estimate of γ_c which is independent of — more or less — arbitrary assumptions about the energy and angular distributions in the c.m.s. The statistical error associated with eq. (12) is

$$\frac{\sigma_{\gamma_c}}{\gamma_c} = \left[\frac{\exp\left[4\sigma^2\right] - 1}{1 + \exp\left[2(\sigma^2 - \bar{x})\right]} \right]^{\frac{1}{2}} (2N)^{-\frac{1}{2}}. \tag{13}$$

In order to gain a rough idea of the numerical values entering eqs. (12) and (13) we may take the data collected in ref. (2) as a whole. The over-all averages are: $\bar{x} \approx 0.2$ and $\sigma^2 \approx 0.47$. Hence $\gamma_c \approx 0.6\Gamma$ and $(\sigma_{\gamma_c}/\gamma_c) \approx N^{-\frac{1}{2}}$. As a typical example for the application of eq. (12) to an individual jet, we have computed all the quantities of interest for the well known HBD-star (3) in which all momenta have been measured:

a) Half angle method: $\gamma_{\frac{1}{2}} = 12.4$; $E \approx 310$ GeV;

b) Castagnoli method: $\gamma_c = 25.5$; $E \approx 1300$ GeV;

c) eq. (12): $\bar{x} = 0.25$, $\sigma^2 = 0.43$, $\Gamma = 10.6$; $\gamma_c = 6.8 \pm 2.2$; $E \approx 92$ GeV.

2. – At the Varenna Conference (1957), EDWARDS *et al.* (7) have pointed out that the dependence (or independence) of the transverse momentum p_T on the c.m.s. angle ψ can be used to discriminate between various theories of meson production. We proceed now to show how such a test can be made by means of eq. (5), or rather by means of a quantity

$$y_f \equiv \frac{\gamma_f}{\gamma_{\frac{1}{2}}} = \left[\frac{1 + \pi_f^{-2}}{1 + \pi_{\frac{1}{2}}^{-2}} \right]^{\frac{1}{2}}, \tag{14}$$

which is independent of γ_c. If the angular distribution in the c.m.s. is continuous at $\psi = 90°$, $\sin \psi_{\frac{1}{2}} = 1$ and eq. (14) allows us to compare π_f with the momentum of particles emitted at right angles to the shower axis. If π_f does not depend upon ψ_f,

(7) B. EDWARDS, J. LOSTY, D. PERKINS and J. REYNOLDS: *Suppl. Nuovo Cimento*, **8**, 722 (1958).

i.e. upon f, we would expect

$$\langle y_f \rangle = 1 = \text{const} . \tag{15}$$

We have computed all values of y_f for a sample of 110 jets registered in the Krakow, Warsaw and Prague cosmic-ray laboratories ([8]) and in our laboratory ([9]). The jets were divided into two classes according to the (Castagnoli) value of γ_c *viz.*: $\gamma_c < 7$, $(\gamma_c)_{av} = 3.7$ and $\gamma_c > 7$, $(\gamma_c)_{av} = 17.8$. Within each class, average values of y_f were computed for different ranges of f. These average values are plotted against f in Figs. 2*a* and 2*b*, for $\gamma_c < 7$ and $\gamma_c > 7$ respectively.

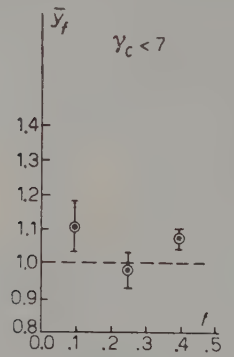

Fig. 2*a*. – Average values of y_f for jets with $\gamma_c < 7$.

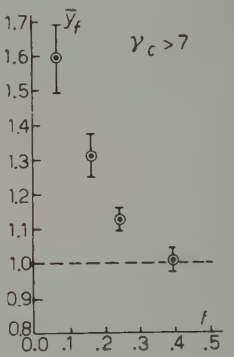

Fig. 2*b*. – Average values of y_f for jets with $\gamma_c > 7$.

As should be expected, in the low energy class eq. (15) is verified, owing to the fact that, most probably, $g_f \approx \gamma_c$ and hence $\gamma_f \approx \gamma_c$ (see eq. (5)). The striking feature of Fig. 1*b* is the significant increase of $\bar{y}_f$ for low values of f, which clearly indicates an increase of π_f (*i.e.* of p_T) with increasing c.m.s. angle θ. This is quite the picture one would expect along the lines of the Fermi theory and it contradicts the predictions of the Landau or Heisenberg theories.

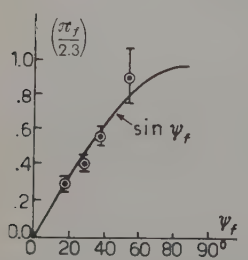

Fig. 3. – Average values of Π_f deduced from eq. (14) for jets with $\gamma_c > 7$, assuming $n = 1$ and $g_0 = 2.5$.

If the angular distribution in the c.m.s. is — as usual — approximated by $\cos^{2n} \psi$, one may assign a definite angle ψ_f to every f: $\psi_f = \arccos (1 - 2f)^{1/(2n+1)}$. A reasonable fit with our data is obtained assuming $n = 1$ and a monoenergetic spectrum: $g_f = g_0 = \text{const}$. Then π_f should vary according to a sine law:

$$\pi_f = (g_0^2 - 1)^{\frac{1}{2}} \sin \psi_f . \tag{16}$$

Fig. 3 is a plot of $\pi_f (g_0^2 - 1)^{-\frac{1}{2}}$ against ψ_f. The full line is the sine function of ψ. The value chosen for g_0 is $g_0 = 2.5$ which corresponds to the average $\bar{x} = 0.2$ for the data of ref. ([2]) (see above, Sect. 1). If a larger value

([8]) P. Ciok, T. Coghen, J. Gierula, J. Pernegr, R. Hołynski, A. Jurak, M. Mięsowicz, T. Saniewska and O. Stanisz: *Nuovo Cimento*, **8**, 166 (1958).

([9]) E. Balea, E. Friedländer and C. Potoceanu: to be published.

of n (*i.e.* a higher degree of anisotropy) is assumed, agreement of the experimental points with the sine law is reached for higher values of g_0. Such an increase of the average particle energy in the c.m.s. with increasing anisotropy is again predicted by Fermi's theory. On the other hand the values of g_0 obtained in this way ($g_0 \sim 2.5$ for $n=1$, $g_0 \sim 5.3$ for $n=5$) are by an order of magnitude lower than the values predicted by Fermi's theory. This would mean that, before disintegrating into distinct particles, the excited meson cloud has cooled off to a temperature of the order of $T \sim \mu$ (see also [10]).

It should be stressed that, independent of the special assumptions made in the preceding paragraph, the strong variation of y_f with f shown by Fig. 1*b* remains as an established experimental fact, which warrants further investigation.

A detailed discussion of the experimental data will be given in a forthcoming paper.

* * *

I am greatly indebted to Prof. M. Mięsowicz, Dr. J. Gierula and Dr. J. Pernegr for kindly making available detailed data on their jets prior to publication, to Prof. J. Ausländer for stimulating discussions and to Miss M. Ionita for help in the numerical computations.

([10]) G. A. Milečin and I. L. Rozental: *Suppl. Nuovo Cimento*, **8**, 770 (1958).

Possible Tests of the Validity of Electrodynamics at Short Distances.

G. Andreassi, P. Budini and I. Reina

Istituto di Fisica dell'Università - Trieste
Istituto Nazionale di Fisica Nucleare - Sottosezione di Trieste

(ricevuto il 24 Aprile 1959)

Some theoretical [1,3] and experimental [4,5] efforts have been devoted recently to the fundamental problem of the limits of validity of electrodynamics at short distances. In the above works the nucleon is used as a known heavy source of virtual photons; this allows the testing of electrodynamic processes at distances that would not be attainable in pure electrodynamic processes, due to the small mass of the electron. The intervention of the nucleus however still remains an unpleasant feature of these works.

In this note we wish to point out that with the machines available now and in the near future the lowering of the present limits of validity of electrodynamics considering some purely electrodynamic processes is possible. In particular the positron-electron annihilation is one of the most favourable ones.

In fact even the total cross section of this process is rather sensitive to form factors introduced in the vertex part of the S-matrix and, considering the unambiguous feature of an event of annihilation, the measurements of the total cross section could be pushed to rather high accuracy using visual or electronic tecniques.

Using the standard methods and standard notations we obtain for the differential cross section in the laboratory system:

$$(1) \qquad \mathrm{d}\sigma^{\mathrm{NL}} = r_0^2\pi \frac{\mathrm{d}\omega_1}{m}\cdot\frac{1}{(\beta\gamma)^2}\left\{\left[\frac{2m}{\omega_2}(1+\gamma) - \frac{m^3(1+\gamma)^2}{\omega_1\omega_2^2} + \frac{\omega_2}{\omega_1}\right]F_1^2 + \right.$$

$$\left. + \left[\frac{2m}{\omega_1}(1+\gamma) - \frac{m^3(1+\gamma)^2}{\omega_1^2\omega_2} + \frac{\omega_1}{\omega_2}\right]F_2^2 + \left[-\frac{2m^2}{\omega_1\omega_2}(1+\gamma)\gamma + \frac{m^4(1+\gamma)^2}{\omega_1^2\omega_2^2}\gamma\right]F_1F_2\right\}.$$

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(1) P. Budini, G. Poiani and I. Reina: *Int. Conf. Padua*, IX, 17 (1957).
(2) S. D. Drell: *Ann. Phys.*, **4**, 75 (1958).
(3) G. Poiani and I. Reina: *Nuovo Cimento*: to be published.
(4) W. K. H. Panofsky: *Ann. Int. Conf. CERN Geneve*, (1958), p. 3.
(5) B. Richter: *Phys. Rev. Lett.*, **3**, 114 (1958).

Assuming as form factors:

$$F_1 = \frac{\Lambda^2}{\Lambda^2 + 2m\omega_1}; \qquad F_2 = \frac{\Lambda^2}{\Lambda^2 + 2m\omega_2}, \tag{2}$$

and putting $\Lambda^{-1} = 0.7\,F$ as the present limits of validity of electrodynamics (2), we obtain for $d\sigma^{NL}/d\sigma^{Loc}$ the graphs of Fig. 1.

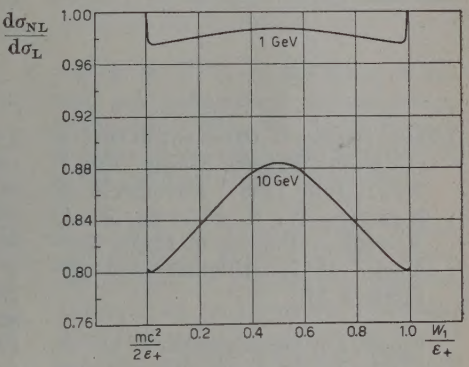

Fig. 1.

With form factors (2) the cross-sections (1) can be integrated exactly. We obtain $\sigma_{tot}^{NL}/\sigma_{tot}^{Loc} = 0.980$ for 1 GeV positrons and $= 0.828$ for 10 GeV positrons.

It can be seen that with 1 GeV positrons the present limits of validity can be lowered by measurements of $\sim 1\%$ accuracy, with 10 GeV positrons [6] by measu rements of $\sim 8\%$ accuracy. These figures are not unreasonable when we consider that the total cross sections at 1 and 10 GeV are 1.06 mb and $1.35 \cdot 10^{-1}$ mb respectively.

Higher order and radiative corrections to these cross sections can be calculated by standard methods and will not alter the above results.

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(6) 10 GeV-positrons could be obtained from the CERN P.S. with a procedure similar to that suggested by G. SALVINI and A. TURRIN: Lab. Naz. di Frascati, Rep. G.27 (1959).

LIBRI RICEVUTI E RECENSIONI

Health Physics - Official Journal of the Health Physics Society, Vol. I, No. 1, June 1958. - Editors: K. Z. MARGAN, W. S. SNYDER, J. A. AUXIER. - Pergamon Press, New York, 1957. Abbonamento annuale $ 17.

Questo è il primo numero di una rivista mensile di « Health Physics » il cui programma è quello di pubblicare lavori che riguardano tutti gli aspetti del problema della protezione dalle radiazioni. La tematica che si affronta nel nuovo ramo della ricerca scientifica è quanto mai varia e va dalla chimica alla fisica, dalla biologia alla medicina, dall'ingegneria alla chimica industriale. Ciò nonostante i vari aspetti del tema della protezione sono strettamente connessi tra loro e il proposito di raccogliere i diversi lavori di specializzazione in una unica rivista, non può che giovare al fine di una completa valutazione del problema.

Il piano di attività della rivista, come risulta sia dalla sua presentazione sia dal carattere degli articoli pubblicati nel suo primo numero, è quello di pubblicare lavori inerenti: *a*) la ricerca pura, *b*) l'ingegneria, *c*) la ricerca applicata, *d*) argomenti di carattere generale. Per lavori di ricerca pura si intendono lavori, sia teorici che sperimentali, che non abbiano immediati fini applicativi riguardanti, ad esempio, il meccanismo dell'azione lesiva delle radiazioni, la misura dei parametri necessari al calcolo della dose o alla progettazione dei dosimetri, gli effetti ecologici delle radiazioni ecc. Per lavori di ingegneria si intendono quelli relativi alla sistemazione dei rifiuti radioattivi, allo sviluppo della strumentazione, alla decontaminazione ecc. Tra i lavori di ricerca applicata sono la dosimetria personale, la dosimetria ambiente gli studi sul fall-out, l'insegnamento nel campo della Fisica Sanitaria. Con lavori generali si comprendono l'organizzazione nazionale ed internazionale, dei servizi di protezione, le questioni inerenti i massimi livelli permessi e così via. Nella rivista sono anche ammessi oltre ai lavori originali lavori di messa a punto e di inquadramento sulle varie questioni.

Dei tredici lavori che compaiono nel primo numero, otto si possono classificare come appartenenti al primo gruppo, e i rimanenti al quarto. Per meglio dare un'idea delle questioni trattate, degli aspetti dei problemi affrontati e del livello dei collaboratori riteniamo conveniente citare i titoli dei diversi lavori dando una breve indicazione del loro contenuto:

1) L. S. TAYLOR (Chairman National Committee on Radiation Protection and Measurements): *Brief History of the National Committee on Radiation Protection and Measurements (NCRP) Covering the Period 1929-1946*. L'articolo illustra l'attività del comitato, le branche in cui è suddivisa la sua organizzazione ed il suo programma di lavoro.

2) W. BRANDT (Du Pont Radiation Physics Laboratory, Wilmington, Del.) *Survey of Stopping Power*. Si discute il qroblema dello « stopping power » per le puestioni inerenti la dosimetria.

3) J. A. Auxier, G.S. Hert and R. E. Zedler (Healt Physics Division, Oak Ridge National Laboratory) *A Single Ion Detector for Maesurement of γ-Ray Ionization in Cavities.* È descritto un nuovo metodo per misurare l'energia assorbita in un mezzo, irradiato con raggi γ.

4) R. D. Birkhoff, H. H. Hubbel jr, J. S. Cheka and R. H. Ritchie (Health Physics Division, Oak Ridge National Laboratory): *Spectral Distribution of Electron Flux in a Beta Radioactive Medium.* Dato un β emettitore (^{32}P) disperso in un mezzo omogeneo (Bakelite) viene calcolato e misurato lo spettro degli elettroni primari e secondari che emergono dalla bakelite in una cavità del mezzo stesso.

5) L. W. Cochran (University of Kentucky): *W for Heavy Particles.* Vengono elencati e discussi i valori di W (energia in eV per la creazione di una coppia di ioni) per diversi gas e miscugli di gas.

6) T. E. Bortner and G. S. Hurt (Oak Ridge National Laboratory): *An apparatus for Measuring Electron Attachement: Results for Oxygen in Argon.* Si descrive un metodo per misurare il grado di adesione elettronica di un gas o di una miscela di gas. Informazioni di questo genere sono utili al fine di stabilire quantitativamente quanto un gas sia adatto al riempimento di un rivelatore. I risultati conducono anche a una migliore comprensione del meccanismo del processo stesso.

7) H. H. Rossi and M. Lubert (Radiological Research Laboratory, Columbia University): *Dosimetry with Ionization Chamber Employing Internal Gas Multiplication.* Viene descritto un particolare tipo di camera a ionizzazione che può essere adatto a misure di dosimetria β.

8) W. S. Snyder (Health Physics Division, Oak Ridge National Laboratory): *Calculation of Radiation Dose.* Si valuta l'influenza della geometria di un fantoccio sul valore della dose e precisamente sul massimo valore della dose che si ha nel fantoccio.

9) W. D. Claus (Division of Biology and Medicine U.S. Atomic Energy Commission): *What is Health Physics?* Articolo di carattere generale sulle finalità del nuovo ramo di ricerca.

10) L. S. Taylor (Chief Atomic and Radiation Physics Division, National Bureau of Standards): *Radiation Exposure as a Reasonable Calculated Risk.* Indagine critica sui criteri con cui vangono stabiliti i massimi livelli di esposizione occupazionale.

11) G. Hoyt Whipple (University of Michigan, School of Public Health): *Health Physics Responsibilities to Management.* Studio che cerca di dare un quadro della responsabilità che l'« health physicst » ha in un centro di ricerche nucleari, sia nell'ambito del centro stesso, sia verso la collettività che sta al di fuori del centro.

12) F. P. Cowan and J. B. H. Kuper (Brookhaven National Laboratory): *Exposure Criteria for Evaluating the Public Consequence of Catastrophic Accidents in Large Nuclear Plants.* Vengono esaminati i vari tipi di esposizione diretta ai prodotti di fissione dovuti a un incidente catastrofico di un reattore e vengono discussi i criteri in base ai quali viene stabilita una dose accettabile di emergenza. Questo lavoro è il sunto di un articolo molto più dettagliato del gruppo del Brookhaven National Laboratory.

13) J. E. Ballou and R. C. Thompson (Biology Operation, Hanford Laboratories): *Metabolism of Cesium* 137 *in*

the Rat: Comparison of Acute and Chronic Administration Experiments. Sono stati condotti già molti lavori sulla valutazione del pericolo e sul metabolismo del ^{137}Cs in diversi animali. Questo lavoro si propone quindi di precisare solo alcuni aspetti particolari del problema.

La varietà dei lavori qui riportati conferma i propositi espressi nella presentazione redazionale. Una iniziativa editoriale come questa non può che giovare a una migliore comprensione del problema e facilita la formazione delle nuove « equipe » di ricercatori (chimici, biologi, ingegneri, fisici e medici) che devono acquistare una comune base di conoscenze in questo campo, completamente nuovo e in continua evoluzione, campo in cui ovviamente l'organizzazione scientifica esistente non ha dietro di sè una lunga tradizione.

O. RIMONDI

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